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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Mathieu Functions of General Order: Connection Formulae, Base Functions and Asymptotic Formulae: V. Approximations in Terms of Higher Transcendental Functions

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MATHIEU FUNCTIONS OF GENERAL ORDER: CONNECTION FORMULAE, BASE FUNCTIONS AND ASYMPTOTIC FORMULAE

V. APPROXIMATIONS IN TERMS OF HIGHER TRANSCENDENTAL FUNCTIONS

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CONTENTS

The methods of part III are further applied to the construction of approximations for the fundamental solution and base functions of part II in terms of higher transcendental functions. The domain of validity is now the complete half-strip $\{z;\ 0 \leqslant \mathrm{Re}\, z \leqslant \frac{1}{2}\pi,\ \mathrm{Im}\, z \geqslant 0\}$ without exceptional point. Relative remainder estimates are again uniformly valid provided they are bounded.

Specifically, approximations are obtained in terms of:

(a) Airy functions, applicable if $\lambda \neq \pm 2h^2$;

(b) parabolic cylinder functions, applicable if $|\lambda| \leqslant 4h^2$, including $\lambda = \pm 2h^2$;

(c) Bessel functions, applicable if $|\lambda| \geqslant 4h^2$; these formulae have maximum relative error $\lambda^{-\frac{3}{2}}h^2 O(1)$ on the half-strip, even if h is arbitrarily small, provided only that λ^{-1} is bounded. This is significantly better when λ/h^2 is large than the corresponding estimate, $\lambda^{-\frac{1}{2}}O(1)$, for the Airy function approximations.

Certain more refined estimates for the auxiliary parameters introduced in part II are also obtained.

1. General

Three types of approximation are considered.

(a) *In terms of Airy functions*

These comprise approximations which remain valid in the neighbourhood of the single transition point on the frontier of the fundamental region Ω (part IV, §1). They do not, however, remain satisfactory when this point is near $z = 0$ or $z = \frac{1}{2}\pi$, that is, when $|\lambda'/h^2| \approx 2$.

(b) *In terms of parabolic cylinder functions*

These are valid for a range of parameters in which a pair of transition points can coalesce, covering the case in which the first type is not applicable. There are two subtypes; the first in terms of ordinary functions, for a range of values of λ'/h^2 including the value -2, for which two transition points coalesce at $z = 0$, and the second, in terms of modified functions, for a range including the value $\lambda'/h^2 = 2$ for which two transition points coalesce at $z = \frac{1}{2}\pi$. The relative order of magnitude of the remainder is $O(h^{-1})$ or smaller, uniformly on the specified domain of variable and parameters. The notation of Miller (1952) is used for the parabolic cylinder functions.

(c) *In terms of Bessel functions*

These take account similarly of a pair of transition points in the upper half-plane. Again there are two subtypes; one in terms of Bessel functions of real order, applicable when $\lambda/h^2 \geqslant c > 2$, and the other in terms of functions of imaginary order, applicable when $\lambda/h^2 \leqslant c < -2$. The relative order of magnitude of the remainder is $\lambda^{-\frac{3}{2}}h^2 O(1)$ or smaller, again uniformly; thus the remainder may be small even if neither parameter is large, in particular if h is small and the order, which is then approximately equal to $\lambda^{\frac{1}{2}}$, is bounded and bounded away from zero. Under these conditions neither elementary function nor Airy function approximations are useful.

The second and third types encompass between them the full range of values of λ'/h^2; it was found convenient to choose specific values of this quotient to delimit the ranges of applicability of the four subtypes, but there is no particular significance in the precise choice made. The methods used in constructing the formulae of these same two types also lead to certain refinements of the estimates for the auxiliary parameters $(\pi\mu, \Phi)$ or $(\hat{\beta}, \Phi)$ introduced in part II.

The calculations required are naturally quite different for the three types of approximation; however, just as in part IV, for any one type they have many common features for different parameter ranges, so that they are only given in detail for one range. However there is one technique used in connection with all three types, based on the use of the maximum modulus principle, to obtain estimates for the error control function in a region containing the relevant transition points.

Expressions for majorant functions for the different basic functions have been determined. They are not included except in the case of Airy functions, since their description is complicated by the fact that they take different forms in different subdomains. Instead, remainder estimates of the form III, (1.5b) have been converted into the form described in part III, §6. As indicated there, the quantity η appearing in each formula may be interpreted as an error relative to the principal term of the formula, except near zeros of the latter where the error is given relative to the amplitude of oscillation of the principal term. In the text, however, as distinct from the tables of formulae, remainder terms have been left in the form II, (1.5b) for compactness; majorant functions are represented by bold-face versions of the symbol for the basic functions, the argument being omitted. Further remarks on the interpretation of remainder terms appear in part I, §2(b).

2. Approximations in terms of Airy functions

With ξ, ξ_0, $F(z)$ defined as in part IV and with

$$\zeta = [\tfrac{3}{2}(\xi-\xi_0)]^{\frac{2}{3}}$$

as independent variable, the Mathieu equation reduces in accordance with III, (3.3) to

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{h^2+\psi_1(z)\}\,\zeta w, \tag{2.1}$$

where
$$y = \zeta^{\frac{1}{4}}F(z)\,w$$

and
$$\psi_1(z) = \psi(z)+\tfrac{5}{36}(\xi-\xi_0)^{-2}, \tag{2.1a}$$

$\psi(z)$ being given by IV, (1.7). The factor $\zeta^{\frac{1}{4}}F(z)$ is analytic and the map $z\to\zeta$ is regular, on the union of the fundamental region Ω and that contiguous fundamental region, depending on the value of λ'/h^2, which shares the transition point z_0 at which $\xi=\xi_0$; $(\xi-\xi_0)^2$ is also analytic on this extended region, although ξ itself is not single valued. The choice of branch for ζ differs in the three cases which arise.

2.1. *The case* $\lambda' < -2h^2$

(*a*) *The variation of the e.c.f.*

By IV, §2.3(c), and (2.1a) above,

$$\psi_1(z) = [\xi-\xi_0]^{-2}\,O(1) \quad \text{on} \quad R_1, \tag{2.2}$$

so that the estimate IV (2.13a) applies also to $\operatorname{var}\limits_{\gamma}\int\psi_1(z)\,\mathrm{d}\xi$ on any path γ in R_1. If $\lambda'\geqslant-4h^2$, $R_1=\Omega$; otherwise, $\Omega=R_1\cup R_2$ and account must be taken of R_2. In the latter case, $[\varDelta(z)]^{-1}=O(1)$ uniformly on R_2, whence by lemma 2 (IV, §2(c))

$$\Psi_2(z) = [\varDelta(z)]^{-\frac{1}{2}}\,O(1)$$

and
$$(\xi-\xi_0)^{-1} = (\xi^*-\xi_0)^{-1}\,O(1) = [\varDelta(z)]^{-\frac{1}{2}}\,O(1).$$

Hence for any family of paths γ in R_2, satisfying the conditions of lemma 1 of part III, §5 for both the functions $\Psi_1(z)$, $\Psi_2(z)$ of IV, §2 (*c*),

$$\operatorname*{var}_{\gamma} \int \psi_1(z)\, d\xi \leqslant \operatorname*{var}_{\gamma} \int \psi(z)\, d\xi + \tfrac{5}{36} \operatorname*{var}_{\gamma} \{[\xi-\xi_0]^{-1}\}$$
$$= \max_{\gamma} \{|\Delta(z)|^{-\frac{1}{2}}\}\, O(1).$$

The derivation of the estimate IV, (3.1) of the e.c.f. for paths of class A is now valid with $\psi(z)$ replaced by $\psi_1(z)$.

However, for present purposes it is necessary to obtain a sharper estimate in the neighbourhood of $z_0 = ia$. Consider therefore the subregion R_4^* of the half-strip $\{z\colon |\mathrm{Re}\, z| \leqslant \frac{1}{2}\pi,\ \mathrm{Im}\, z \geqslant 0\}$ defined by $|\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a$. By III, §3, $\psi_1(z)\,(\xi-\xi_0)^{\frac{2}{3}}$ is analytic on R_4^*, which contains z_0 in its interior; hence on R_4^*, by the maximum modulus principle,

$$|\psi_1(z)| \leqslant M|\xi-\xi_0|^{-\frac{2}{3}}, \tag{2.3}$$

where
$$M = \sup\{|\psi_1(z)\,(\xi-\xi_0)^{\frac{2}{3}}|\colon z \in \mathrm{Fr}\, R_4^*\}.$$

Integrating this, for any path γ in R_4^*, gives

$$\operatorname*{var}_{\gamma} \int \psi_1(z)\, d\xi \leqslant \operatorname*{var}_{\gamma} \Psi_4(z)$$

where
$$\Psi_4(z) = 3M(\xi-\xi_0)^{\frac{1}{3}}.$$

Now let z, z^* be frontier points of R_4^* and let ξ^* be the value of ξ corresponding to the latter. Then from IV, (2.1) or IV, (2.8),

$$(\xi-\xi_0)^{\pm 1} = (\xi^*-\xi_0)^{\pm 1}\, O(1), \tag{2.4}$$

whence by (2.2) above, which is valid on R_4^*,

$$\psi_1(z) = (\xi^*-\xi_0)^{-2}\, O(1).$$

Combining these results gives

$$M = (\xi^*-\xi_0)^{-\frac{4}{3}}\, O(1).$$

Finally, if z is replaced by an arbitrary point of R_4, (2.4) with the positive sign remains valid, whence

$$\Psi_4(z) = (\xi^*-\xi_0)^{-1}\, O(1)$$
$$= \begin{cases} (\sinh a)^{-1}\, O(1) & \text{if} \quad \sinh a \geqslant 1, \\ (\sinh a)^{-2}\, O(1) & \text{if} \quad \sinh a \leqslant 1. \end{cases} \tag{2.5}$$

There now exist paths to supplement class A (IV, §3), originating from $\infty\mathrm{i}$ and from $-\frac{1}{2}\pi+\infty\mathrm{i}$, being $+\xi$-progressive, terminating at an arbitrary point of R_4, and satisfying the conditions of lemma 1 of part III and its corollary. The formula (2.5) provides an estimate of the variation of the e.c.f. on these paths; for paths of class A not terminating in a point of R_4, it has already been seen that IV, (3.1) is valid.

(*b*) *Asymptotic formulae*

With the branch of ζ determined as in §2.4 (*a*), the formulae (2.12 *a*, *b*) can now be established, the domain of validity of the former being provisionally restricted to the region

$$\{z \in \Omega\colon \arg(\xi-\xi_0) \geqslant -\tfrac{1}{4}\pi\} \cup R_4.$$

The constant factors are obtained by substituting the appropriate argument in place of x in the asymptotic formula (2.8) and by comparing the result with IV, (1.9) or IV, (1.10). A further formula, valid on

$$\{z \in \Omega \colon \arg(\xi - \xi_0) \leqslant -\tfrac{1}{4}\pi\} \cup R_4,$$

is

$$y_1(z+\pi) = 2\pi^{\frac{1}{2}} \mathrm{e}^{hE_1} h^{\frac{1}{6}} [\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \zeta^{\frac{1}{4}}] \{\mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} \operatorname{Ai}(h^{\frac{2}{3}} \zeta \mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}) + \mathbf{Ai}\, \eta\}, \tag{2.6}$$

which also has the remainder estimate (2.13), and whose form is the complex conjugate of (2.12b).

This formula and (2.12b) can be used, with the aid of the connection formulae II, (4.1.2) and (2.9a) below, to establish the validity of (2.12a) on the remainder of Ω. The method is similar to that used in IV, §4(c) to derive the formulae IV, (6.3.4); in the same way, by using the definition of $y_2(\mathrm{i}x)$ (II, (4.2.2)) and the connection formula (2.9b) below, (2.14) is obtained. The majorant of the Airy function on the left-hand side of (4.9a, b) is in each case, on the relevant domain, the sum of the majorants of those on the right-hand side.

2.2. *Other parameter ranges*

Methods similar to those of §2.1 are used throughout, and only the case $-2h^2 < \lambda' < 2h^2$ will be considered in any detail. Here, the transition point lies in $(0, \frac{1}{2}\pi)$ and the appropriate region for the estimation of the e.c.f. is the strip $\{z\colon 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\pi\}$. The conclusion reached is that the formula A* (IV, (5.1)) remains valid under the same conditions, but that with a class of paths supplemented as in §2.1 above, the sharper result

$$\operatorname*{var}_{\gamma} \int \psi_1(z)\, \mathrm{d}\xi = (\sin 2a)^{-2}\, O(1)$$

applies if $z \in R_4 = \{z \in \Omega\colon |\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a\}$.

With the branch of ζ determined as in §2.4(b), the formula (2.15a, b) can now be derived, together with the formula

$$y_1(-z) = 2\pi^{\frac{1}{2}} \mathrm{e}^{-hE_1} h^{\frac{1}{6}} [F(z)\, \zeta^{\frac{1}{4}}]\, \mathrm{e}^{-\mathrm{i}(\frac{1}{4}\pi + hE)} \{\mathrm{e}^{\frac{1}{6}\mathrm{i}\pi} \operatorname{Ai}(h^{\frac{2}{3}} \zeta \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}) + \mathbf{Ai}\, \eta\}, \tag{2.7}$$

valid on $\{z \in \Omega\colon \arg(\xi - \xi_0) \geqslant \frac{5}{4}\pi\} \cup R_4^*$, with the same remainder estimate (2.16) (see figure 4, part IV). This formula is complex conjugate in form to (2.15a). By a method similar to that used in IV, §4(b) to obtain IV, (6.3.1c), the modified form (2.15c) of (2.7), valid on $\{z \in \Omega\colon \arg(\xi - \xi_0) \geqslant \frac{3}{4}\pi\}$, can be derived.

Now define

$$\phi = \Phi - (\tfrac{1}{4}\pi + hE),$$

estimates for which are given by IV, (6.2.4) and by (3.26), (4.14), (4.17) below. Then (2.15a) and (2.7), which is also valid on $[0, \frac{1}{2}\pi]$, together with the defining formula II, (4.3.6) for $y_3(x)$ and the connection formulae (2.9a, b), give (2.17a). Similarly, but by using II, (3.6), (2.17b) is obtained; this formula is not valid on $(a, \frac{1}{2}\pi]$ since $y_3(\pi - x)$ is recessive there. On this interval, however, $\pi - x$ may be substituted for x in II, (4.3.6), and formula (2.15b) and its conjugate used to estimate the result; this gives (2.17c). These three formulae are complicated; however, they can be simplified by replacing ϕ by zero, with some loss of precision unless $[\cos a]^{-1}$ is bounded. Indeed with this last condition, it follows from IV, (6.2.4) that the effect of this substitution can be absorbed into the remainder term.

No new methods are needed for the case $\lambda' > 2h^2$.

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2.3. *Properties of Airy functions*

These functions satisfy the differential equation

$$y'' = xy.$$

The standard solution, characterized by its behaviour as $x \to \infty$ with $|\arg x| \leqslant \frac{1}{3}\pi$, is

$$\mathrm{Ai}\,(x) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\mathrm{e}^{-t}, \tag{2.8}$$

where $t = \frac{2}{3}x^{\frac{3}{2}}$ takes its principal value on $|\arg x| < \pi$. The formula (2.8) is valid as $x \to \infty$ with $|\arg x| \leqslant \pi - \delta$ $(\delta > 0)$. A second solution $\mathrm{Bi}\,(x)$, which with $\mathrm{Ai}\,(x)$ satisfies Miller's criteria, is defined by (2.9*b*) below. Other solutions are $\mathrm{Ai}\,(x\mathrm{e}^{\pm\frac{2}{3}\mathrm{i}\pi})$.

(*a*) *Connection formulae*

$$\mathrm{Ai}\,(x) = \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi}\,\mathrm{Ai}\,(x\mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}) + \mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi}\,\mathrm{Ai}\,(x\mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}), \tag{2.9a}$$

$$\mathrm{Bi}\,(x) = -\mathrm{i}\,\mathrm{e}^{\frac{1}{3}\mathrm{i}\pi}\,\mathrm{Ai}\,(x\mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}) + \mathrm{i}\,\mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi}\,\mathrm{Ai}\,(x\mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}). \tag{2.9b}$$

(*b*) *Majorant functions*

Defining $\Theta(x) = \min\,\{1.12|x|^{-\frac{1}{4}}, 1.29\}$ gives

$$|\mathrm{Ai}\,(x)| \leqslant \mathbf{Ai}\,(x) = \begin{cases} \frac{1}{2}\pi^{-\frac{1}{2}}|\mathrm{e}^{-t}|\Theta(x) & \text{if} \quad |\arg x| \leqslant \frac{2}{3}\pi, \\ \frac{1}{2}\pi^{-\frac{1}{2}}\{|\mathrm{e}^{-t}| + |\mathrm{e}^{t}|\}\,\Theta(x) & \text{if} \quad |\arg(-x)| \leqslant \frac{1}{3}\pi, \end{cases} \tag{2.10}$$

$$|\mathrm{Bi}\,(x)| \leqslant \mathbf{Bi}\,(x) = \begin{cases} \pi^{-\frac{1}{2}}|\mathrm{e}^{t}|\Theta(x) & \text{if} \quad |\arg x| \leqslant \frac{2}{3}\pi, \\ \frac{1}{2}\pi^{-\frac{1}{2}}\{|\mathrm{e}^{-t}| + |\mathrm{e}^{t}|\}\,\Theta(x) & \text{if} \quad |\arg(-x)| \leqslant \frac{1}{3}\pi. \end{cases} \tag{2.11}$$

If x is real and negative, $|\mathrm{e}^{-t}| = |\mathrm{e}^{t}| = 1$.

2.4. *Table of asymptotic formulae*

Except for the variable ζ introduced in each case below, the notation, and in particular the interpretation of the expression $O(1)$, follows that of part II, §4 and part IV, §6.

Majorant functions for $\mathrm{Ai}\,(x)$, $\mathrm{Bi}\,(x)$ are denoted by **Ai**, **Bi** respectively (see §2.3 (*b*)); the argument is omitted, being in each formula the same as for the Airy function itself.

(*a*) *The case* $\lambda' < -2h^2$

On Ω define $\zeta = [\frac{3}{2}(\xi - \xi_0)]^{\frac{2}{3}}$, with $\arg \zeta = 0$ when $\arg(\xi - \xi_0) = \pi$ (see figure 3, part IV), so that ζ is real and positive if $\mathrm{Re}\, z = 0$ and $\mathrm{Im}\, z > a$.

(i) *Complex variable.* On Ω,

$$y_1(z) = 2\pi^{\frac{1}{2}}\mathrm{e}^{-hE_1}h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\,\zeta^{\frac{1}{4}}]\,\{\mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\}, \tag{2.12a}$$

$$y_1(z-\pi) = 2\pi^{\frac{1}{2}}\mathrm{e}^{hE_1}h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\,\zeta^{\frac{1}{4}}]\,\{\mathrm{e}^{-\frac{1}{3}\mathrm{i}\pi}\,\mathrm{Ai}\,(h^{\frac{2}{3}}\zeta\mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}) + \mathbf{Ai}\,\eta\}, \tag{2.12b}$$

where, uniformly,

$$\eta = h^{-1}O(1) \times \begin{cases} \max\,\{(\sinh a)^{-1}, (\sinh a)^{-2}\} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a \\ (\Delta(z))^{-1} & \text{if} \quad \frac{1}{2}\sinh^2 a \leqslant |\Delta(z)| \leqslant 1 \\ (\Delta(z))^{-\frac{1}{2}} & \text{otherwise.} \end{cases} \tag{2.13}$$

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(ii) *Modified equation*, $q < 0$. If $x \geqslant 0$, then with $z = \mathrm{i}x$, $y_1(\mathrm{i}x)$ is given by (2.12*a*) and

$$y_2(\mathrm{i}x) = \pi^{\frac{1}{2}}(\sinh \pi\mu)^{-1}\, \mathrm{e}^{hE_1} h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \zeta^{\frac{1}{4}}]\, \{\mathrm{Bi}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Bi}\,\eta\}. \tag{2.14}$$

The factor in square brackets is real and positive, and η satisfies (2.13).

(*b*) *The case* $-2h^2 < \lambda' < 2h^2$

On Ω define $\zeta = [\frac{3}{2}(\xi - \xi_0)]^{\frac{2}{3}}$, with $\arg \zeta = 0$, when $\arg(\xi - \xi_0) = 0$ (see figure 4, part IV), so that ζ is real and positive if $z \in (a, \frac{1}{2}\pi]$.

(i) *Complex variable*. On Ω,

$$y_1(z) = 2\pi^{\frac{1}{2}}\, \mathrm{e}^{-hE_1} h^{\frac{1}{6}}\, \mathrm{e}^{\mathrm{i}(\frac{1}{4}\pi + hE)}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\mathrm{e}^{-\frac{1}{6}\mathrm{i}\pi}\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta\, \mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}) + \mathbf{Ai}\,\eta\}, \tag{2.15a}$$

$$y_1(z - \pi) = 2\pi^{\frac{1}{2}}\, \mathrm{e}^{hE_1} h^{\frac{1}{6}}\, \mathrm{e}^{-\mathrm{i}(\frac{1}{4}\pi + hE)}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad \mathrm{Im}\, \xi \geqslant 0, \tag{2.15b}$$

$$y_1(-z) = 2\pi^{\frac{1}{2}} \hat{\beta}\, \mathrm{e}^{hE_1} h^{\frac{1}{6}}\, \mathrm{e}^{-\mathrm{i}(\frac{1}{4}\pi + hE)}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\mathrm{e}^{\frac{1}{6}\mathrm{i}\pi}\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta\, \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad \mathrm{Im}\, \xi \leqslant 0, \tag{2.15c}$$

where, uniformly,

$$\eta = h^{-1} O(1) \times \begin{cases} [\sin 2a]^{-2} & \text{if} \quad |\varDelta(z)| \leqslant \frac{1}{8} \sin^2 2a, \\ [\varDelta(z)]^{-1} & \text{if} \quad \frac{1}{8}\sin^2 2a \leqslant |\varDelta(z)| \leqslant 1, \\ [\varDelta(z)]^{-\frac{1}{2}} & \text{otherwise}. \end{cases} \tag{2.16}$$

(ii) *Ordinary equation*, $q < 0$. If $x \in [0, \frac{1}{2}\pi]$ then with $z = x$,

$$y_3(x) = \pi^{\frac{1}{2}}\, \mathrm{e}^{-hE_1} h^{\frac{1}{6}}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\cos\phi\, \mathrm{Bi}\,(h^{\frac{2}{3}}\zeta) - \sin\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Bi}\,\eta\}, \tag{2.17a}$$

$$y_3(\pi - x) = \pi^{\frac{1}{2}} \beta^*\, \mathrm{e}^{-hE_1} h^{\frac{1}{6}}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\sin\phi\, \mathrm{Bi}\,(h^{\frac{2}{3}}\zeta) + \cos\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad x \leqslant a, \tag{2.17b}$$

$$y_3(\pi - x) = 2\pi^{\frac{1}{2}}\, \mathrm{e}^{hE_1} h^{\frac{1}{6}}[F(z)\, \zeta^{\frac{1}{4}}]\, \{\cos\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad x \geqslant a, \tag{2.17c}$$

where η satisfies (2.16), $\quad \beta^* = (1 + |\beta|)/\beta,$

and

$$\phi = \varPhi - (\tfrac{1}{4}\pi + hE) \tag{2.18}$$

may be estimated by means of IV, (6.2.4) or V, (3.26) or (4.14, 17). The factor in square brackets is real and positive.

These formulae may be simplified by replacing ϕ by zero, but there is some loss of precision unless $(\cos a)^{-1}$ is bounded.

(iii) *Ordinary equation*, $q > 0$. Formulae for $y_3(\frac{1}{2}\pi \pm x)$ valid on $[0, \frac{1}{2}\pi]$ are obtained by substituting $\frac{1}{2}\pi - x$ for x and for z in (2.17*a*, *b*, *c*).

(*c*) *The case* $\lambda' > 2h^2$

On Ω define ζ as in (*b*) above, so that ζ is real and positive if $z \in [\frac{1}{2}\pi, \frac{1}{2}\pi + \mathrm{i}a]$ (see figure 5, part IV).

(i) *Complex variable*. On Ω the formulae (2.15*a*, *b*, *c*) hold but with η satisfying (2.13).

(ii) *Modified equation*, $q > 0$. On $x \geqslant 0$, with $z = \frac{1}{2}\pi + \mathrm{i}x$,

$$y_4(x) = \pi^{\frac{1}{2}} h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \zeta^{\frac{1}{4}}]\, \{\cos\phi\, \mathrm{Bi}\,(h^{\frac{2}{3}}\zeta) + \sin\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Bi}\,\eta\}, \tag{2.19a}$$

$$y_4(-x) = \pi^{\frac{1}{2}} \beta^\dagger h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \zeta^{\frac{1}{4}}]\, \{-\sin\phi\, \mathrm{Bi}\,(h^{\frac{2}{3}}\zeta) + \cos\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad x \geqslant a,$$

$$\text{where } \beta^\dagger = |\beta| + \hat{\beta}, \tag{2.19b}$$

$$y_4(-x) = 2\pi^{\frac{1}{2}}\, \mathrm{e}^{-2hE_1} h^{\frac{1}{6}}[\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \zeta^{\frac{1}{4}}]\, \{\cos\phi\, \mathrm{Ai}\,(h^{\frac{2}{3}}\zeta) + \mathbf{Ai}\,\eta\} \quad \text{if} \quad x \leqslant a, \tag{2.19c}$$

where η satisfies (2.13) and ϕ is defined by (2.18). The factor in square brackets is real and positive. Simplified formulae are obtained by replacing ϕ by zero, but there is some loss of precision unless $|\Delta(z)| = \sinh^2 a\, O(1)$.

3. Approximations in terms of parabolic cylinder functions

3.1. *The parameter range* $-4h^2 \leqslant \lambda' \leqslant 0$

(*a*) *The Liouville transformation*

The basic equation used is III, (1.9) with the positive sign and with $u = h$, so that, reference being made to IV, (1.3) for the definition of ξ,

$$\frac{d\zeta}{dz} = \frac{4[\Delta(z)]^{\frac{1}{2}}}{(\zeta^2-\alpha)^{\frac{1}{2}}} = 2\left\{\frac{\lambda'/h^2+2\cos 2z}{\alpha-\zeta^2}\right\}^{\frac{1}{2}}. \tag{3.1}$$

If $\lambda' \geqslant -2h^2$, the transition points in the strip $\{z: |\text{Re}\, z| \leqslant \frac{1}{2}\pi\}$ are at $z = \pm a$ (see IV, §6 for notation); since these points are required to map into $\zeta = \pm\alpha^{\frac{1}{2}}$, then by III, (1.9*a*),

$$\alpha = \frac{4}{\pi}\int_{-a}^{a} [\Delta(z)]^{\frac{1}{2}}\, dz = \frac{8E}{\pi}. \tag{3.1a}$$

This formula is also valid if $\lambda' < -2h^2$, but E and thus also α are then negative.

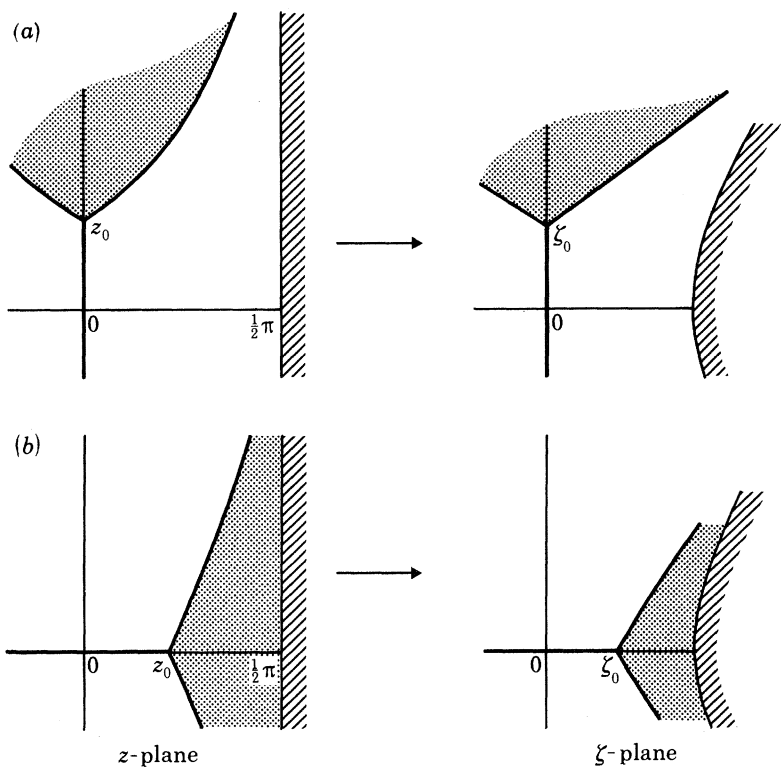

Figure 1. The map $z \to \zeta$: $-4h^2 \leqslant \lambda' \leqslant 0$; (*a*) $\lambda' < -2h^2$; (*b*) $\lambda' > -2h^2$.

Since $\Delta(z)$ is an even function, $\zeta = 0$ at $z = 0$; the sign of the square root in (3.1) will be determined so that $d\zeta/dz$ is positive at $z = 0$, when it is also real and positive on both real and imaginary axes within the above strip. The map $z \to \zeta$ is one-to-one analytic on the strip and is illustrated in figure 1, the two cases $\lambda' \lessgtr -2h^2$, corresponding to $\alpha \lessgtr 0$ being shown separately. The principal curves are indicated in both z- and ζ-planes, z_0 being defined as in part IV, §2 or §6.1, with ζ_0 as the corresponding value of ζ; the stippled regions are shadow zones for certain classes of paths, as indicated later.

The Mathieu equation II, (1.4) is transformed to

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = (\zeta^2-\alpha)\,\{h^2+\psi_2(z)\}\,w,$$

where $\psi_2(z)$ is given by III, (3.4*a*) and $\psi(z)$ by IV, (1.7*b*), while

$$y = G(z)\,w,$$

where

$$G(z) = F(z)\,(\zeta^2-\alpha)^{\frac{1}{4}} = \left[\frac{(\zeta^2-\alpha)}{\Delta(z)}\right]^{\frac{1}{4}} = \left[\frac{1}{4}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right]^{-\frac{1}{2}};$$

the factor $G(z)$ is regular and without zeros on the same strip, and the branch of the fractional power will be chosen so that $G(z)$ is real and positive on both axes.

(*b*) *Estimation of the e.c.f.*

The fundamental region Ω maps *into* the first quadrant in the ζ-plane, and it can be shown by means of III, (1.9*b*) that on Ω,

$$[\xi-\xi_0]^{\pm 1} = \begin{cases}[\zeta^2-\alpha]^{\pm 1}\,O(1) & \text{if } |\zeta^2-\alpha| \geqslant |\alpha|,\\ \alpha^{\mp\frac{1}{2}}[\zeta^2-\alpha]^{\pm\frac{3}{2}}\,O(1) & \text{if } |\zeta^2-\alpha| \leqslant |\alpha|;\end{cases} \tag{3.2}$$

the proof is similar to the derivation of properties of the function f introduced in IV, §2.2. From (3.2) and III, (3.4*a*), it follows that

$$\psi_2(z)-\psi(z) = 3(\zeta^2-\alpha)^{-2}+5(\zeta^2-\alpha)^{-3} = [\xi-\xi_0]^{-2}\,O(1)$$

uniformly. Hence also, by IV, (2.7), which is valid on the whole of Ω for the parameter range under consideration,

$$\psi_2(z) = [\xi-\xi_0]^{-2}\,O(1), \tag{3.3}$$

so that

$$\operatorname{var}\int \psi_2(z)\,\mathrm{d}\xi = \operatorname{var}\{(\xi-\xi_0)^{-1}\}\,O(1), \tag{3.3a}$$

corresponding to IV, (2.13*a*), and the results of IV, §§2, 3 are valid for ψ_2.

The required sharper estimate can be obtained on the fixed subregion

$$R_5 = \{z \in \Omega\colon |\sin z| \leqslant 1\}.$$

Since $|\sin z_0| \leqslant \frac{1}{2}$, $|\Delta(z)|^{-1} = O(1)$ uniformly when $|\sin z| = 1$, whence from IV, (2.8) and from (3.2), (3.3) above,

$$(\zeta^2-\alpha)\,\psi_2(z) = O(1) \tag{3.4}$$

uniformly on the portion of the curve $\{z\colon |\sin z| = 1\}$ which lies in Ω, and hence by symmetry on Fr R_5^*, where R_5^* is the union of R_5 and its three successive reflexions in the axes. But according to III, §3, this function is analytic on R_5^*, whence by the maximum modulus principle (3.4) is uniformly valid on R_5^*.

Integrating gives

$$\operatorname{var} \int \psi_2(z)\, d\xi = \operatorname{var} \{\arcsin(\zeta\alpha^{-\frac{1}{2}})\}\, O(1)$$

on any path in R_5^*. This is not satisfactory, however, since for fixed $\zeta \neq 0$, arcsin $\zeta\alpha^{\frac{1}{2}}$ is of the order of $|\ln \alpha|$, so that its variation does not remain bounded as $\alpha \to 0$; the following device overcomes the difficulty. The function

$$\zeta^{-1}\{(\zeta^2 - \alpha)\, \psi_2(z) + \alpha\psi_2(0)\}$$

is also analytic on R_5^* and is uniformly bounded on its frontier; it is therefore uniformly bounded on R_5^*. It follows that

$$\psi_2(z)\, d\xi/d\zeta = (\zeta^2 - \alpha)^{-\frac{1}{2}}\, \{\zeta\, O(1) - \alpha\psi_2(0)\},$$

whence

$$\int \psi_2(z)\, d\xi = O(1) + \alpha \arcsin(\zeta\alpha^{-\frac{1}{2}})\psi_2(0).$$

The next step is to calculate $\psi_2(0)$; this can readily be done by means of IV, (1.7*b*) and III, (3.4*a*), which give

$$\psi_2(0) = -\tfrac{1}{4}(\lambda - \lambda')/\Delta(0) + \tfrac{1}{8}[\Delta(0)]^{-2} - 2\alpha^{-2}.$$

Expansion of the integral E in powers of $\Delta(0)$ gives

$$\alpha = 8E/\pi = -4\Delta(0) + \tfrac{1}{2}[\Delta(0)]^2 + [\Delta(0)]^3\, O(1),$$

leading to

$$\psi_2(0) = (\lambda - \lambda' + \tfrac{1}{8})\, \alpha^{-1} + O(1).$$

Thus,

$$\int_{\xi=0} \psi_2(z)\, d\xi = O(1) \tag{3.5}$$

on R_5^* provided that λ' is determined so that

$$\lambda' - \lambda = \tfrac{1}{8} + |\ln \alpha|^{-1}\, O(1), \tag{3.5a}$$

the natural choice being $\lambda' - \lambda = \frac{1}{8}$.

By means of (3.3*a*) and (3.5), the paths of class A introduced in part IV can now be supplemented (cf. § 2.1 (*a*) above) by paths γ originating from infinity in the same manner and terminating in an arbitrary point of R_5.

For such paths,

$$\operatorname*{var}_{\gamma} \int \psi_2(z)\, d\xi = O(1).$$

In both the cases $\lambda' \lessgtr -2h^2$, the domain accessible from either ∞ i or $\frac{1}{2}\pi + \infty$ i is the whole of Ω. If $|\sin z| \geqslant 1$, $|\Delta(z)|^{-1}$ is bounded, and hence for the extended class of paths, by means of IV, (3.1) or IV, (5.1) together with the above result,

$$\operatorname*{var}_{\gamma} \int \psi_2(z)\, d\xi = \begin{cases} [\Delta(z)]^{-\frac{1}{2}}\, O(1) & \text{if} \quad |\sin z| \geqslant 1, \\ O(1) & \text{otherwise.} \end{cases} \tag{3.6a}$$

For suitable progressive paths originating from $-\frac{1}{2}\pi + \infty$ i or from $-\infty$ i and terminating in an arbitrary point of Ω, apart from certain shadow zones, or at $\frac{1}{2}\pi + \infty$ i or ∞ i respectively,

$$\operatorname{var} \int \psi_2(z)\, d\xi = O(1). \tag{3.6b}$$

In the first of these two cases, (3.6 *a*) in fact applies if $\arg(\xi-\xi_0) \geqslant \frac{3}{4}\pi$; in neither case is it valid elsewhere since the path must pass through the region R_5^*.

The shadow zones, not accessible by ξ-progressive paths, arise for paths originating from $-\frac{1}{2}\pi+\infty\mathrm{i}$ if $\lambda' > -2h^2$, and from $-\infty\mathrm{i}$ if $\lambda' < -2h^2$; they are illustrated in figure 1.

(*c*) *Asymptotic formulae*

On introducing the notation $\kappa = 2hE/\pi = \frac{1}{4}\alpha h,$

the appropriate substitutions into the asymptotic formula (3.13) are

$$\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}\zeta h^{\frac{1}{2}},\ \kappa,\ -h(\xi+E_1) \quad \text{for} \quad x,\ a,\ t$$

respectively; the last of these three depends on the conditions (3.12 *a*) and on the values of ξ_0 or of $\xi(0)$ according to the sign of κ (see figures 3, 4, part IV). The result is

$$U(\kappa,\ -\mathrm{i}\zeta h^{\frac{1}{2}}) \sim \mathrm{e}^{\frac{1}{2}\kappa}|\kappa|^{-\frac{1}{2}\kappa}[\tfrac{1}{4}h(\zeta^2-\alpha)]^{-\frac{1}{4}}\mathrm{e}^{h(\xi+E_1)}$$

as $z \to \infty\mathrm{i}$, the parabolic cylinder function being a solution of the basic equation III, (1.9). Comparison with IV, (1.9) shows that the corresponding solution of the Mathieu equation is a multiple of $y_1(z)$, the constant factor being determined at the same time. This leads to the formula (3.20 *a*), the remainder estimate being a consequence of (3.6 *a*); the companion formula (3.20 *b*) is obtained similarly.

If $z \to \frac{1}{2}\pi+\infty\mathrm{i}$ with $\mathrm{Re}\, z = \frac{1}{2}\pi$, the exponential approximations required by the method of III, §4 for the estimation of connection coefficients are available from part IV. If $\kappa \leqslant 0$, the formula

$$y_1(z+\pi) = \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+\frac{1}{2}\kappa)}\mathrm{e}^{hE_1}(|\kappa|/\mathrm{e})^{-\frac{1}{2}\kappa}h^{\frac{1}{4}}G(z)\,\{U(-\kappa,\ -h^{\frac{1}{2}}\zeta)+\boldsymbol{U}\,\eta\}, \tag{3.7}$$

which in form is complex conjugate to (3.20 *b*), but with $-\zeta$ in place of ζ, is valid when $\mathrm{Re}\, z = \frac{1}{2}\pi$ with $\eta = h^{-1}\,O(1)$; with (3.20 *a*) and the connection formula (3.11 *a*), this leads to (3.18) as a refinement of IV, (6.2.1). If $\kappa \geqslant 0$, a corresponding formula for $y_1(-z)$ leads to the refinement (3.19) of IV, (6.2.3). No improvement is possible for the estimates IV, (6.2.2), (6.2.4) for the phase parameter Φ, though in the parameter range considered here the remainder term is always $h^{-1}O(1)$.

The construction of formulae for real bases again calls for separate consideration of the cases $\kappa \gtrless 0$.

When $\kappa \leqslant 0$, and $z = \mathrm{i}x$ $(x \geqslant 0)$, (3.7) is valid with η given, as in (3.20 *b*), by (3.21); the connection formulae (3.11 *b*) and II, (4.2.2) give the formula (3.22) for $y_2(\mathrm{i}x)$. For the ordinary equation, the formula (3.23) for the characteristic solutions can be found by the method used in IV, §4 (*c*), but by using the formulae (3.7) and (3.20 *b*) for $y_1(z\pm\pi)$.

When $\kappa \geqslant 0$, the formulae given in §3.4 for $y_1(-\mathrm{i}x)$ and for $y_3(x)$ and $y_3(\pi-x)$ are obtained by similar methods.

In neither case can the formulae of part IV for modified functions with $q < 0$ be improved on.

(*d*) *A lemma*

The following is required in part IV, §4 (*b*).

Lemma 1. *Let* $\delta > 0$. *Then there is a constant* $k > 0$ *such that provided* $|\lambda| > k$,

$$\text{(i)}\quad |\lambda+2h^2| \leqslant \delta \Rightarrow c_0(q) < \lambda < a_0(q);$$

$$\text{(ii)}\quad \lambda+2h^2 \leqslant \delta \Rightarrow \mathrm{e}^{-\pi\mu} = |\lambda|^{-\frac{1}{2}}\,O(1) \quad \textit{uniformly}.$$

Proof. The symbol k will be used to denote a generic constant, to which in each case a suitable value must be assigned; it is further assumed whenever necessary that $|\lambda| > k$. Define λ' in accordance with (3.5*a*) by $\lambda' - \lambda = \frac{1}{8}$, and without loss of generality suppose $\delta \geqslant \frac{1}{8}$.

If $|\lambda + 2h^2| \leqslant \delta$, then a simple estimate gives

$$hE = [(\lambda' + 2h^2)/h]\, O(1),$$

whence by IV, (4.6), with $n = 0$,

$$\arg\beta = \tfrac{1}{2}\pi + O(h^{-1}). \tag{3.8}$$

Hence if $h > k$, then $\arg\beta > 0$, $\lambda > c_0(q)$ and II, §4.3 is applicable.

If $\lambda + 2h^2 \leqslant \delta$, $\lambda' + 2h^2 > 0$ and $h > k$, then by (3.19)

$$\hat{\beta} = e^{-2hE_1}\, O(1); \tag{3.9}$$

hence $\arctan\hat{\beta} < \frac{1}{4}\pi$, and by (3.8), $\arg\beta < \pi - \arctan\hat{\beta}$ and from table 2 of part II, §4.3, $\lambda < a_0(q)$.

If $\lambda' + 2h^2 \leqslant 0$, then $\lambda = \lambda' - \frac{1}{8} < -2h^2 < a_0(q)$, which completes the proof of the first part of lemma 1.

Next, with the same conditions as for (3.9), by II, (4.1.5), $\cosh(\pi\mu) = (\hat{\beta}^{-1} + \hat{\beta}) \sin\arg\beta$, whence by (3.8), (3.9), $(\cosh\pi\mu)^{-1} = e^{-2hE_1}\, O(1)$.

If $\lambda' + 2h^2 \leqslant O$, but with $|\lambda| > k$ only, in place of $h > k$, the same conclusion follows from IV, (6.2.1).

Finally, over the complete range $\lambda + 2h^2 \leqslant \delta < a_0(q)$ we have $e^{-\pi\mu} = (\cosh\pi\mu)^{-1}\, O(1)$ and another simple estimate gives $[hE_1]^{-1} = |\lambda|^{-\frac{1}{2}}\, O(1)$, whence the second part of the lemma follows without difficulty.

3.2. *The parameter range* $0 \leqslant \lambda' \leqslant 4h^2$

It is now more convenient to define ζ by

$$\frac{d\zeta}{dz} = \frac{4[\Delta(z)]^{\frac{1}{2}}}{(\alpha - \zeta^2)^{\frac{1}{2}}} = 2\left\{\frac{\lambda'/h^2 + 2\cos 2z}{\zeta^2 - \alpha}\right\}^{\frac{1}{2}},$$

in place of (3.1), where now $\alpha = 8E_1/\pi$

and $\zeta = 0$, $d\zeta/dz > 0$ when $z = \frac{1}{2}\pi$. The relevant transition points are now at $z = z_0$, $\pi - z_0$ in $[\frac{1}{4}\pi, \frac{3}{4}\pi]$ if $\lambda' \leqslant 2h^2$ and on the line $\{z\colon \operatorname{Re} z = \frac{1}{2}\pi\}$ if $\lambda' \geqslant 2h^2$. The map $z \to \zeta$ is one-to-one analytic on the strip $\{z\colon 0 \leqslant \operatorname{Re} z \leqslant \pi\}$ and is illustrated in figure 2, with principal curves, in the separate cases $\lambda' \lesseqgtr 2h^2$.

The Mathieu equation transforms into

$$d^2w/d\zeta^2 = (\alpha - \zeta^2)\,\{h^2 + \psi_2(z)\}\, w,$$

where $$y = G(z)\, w$$

with $$G(z) = [(\alpha - \zeta^2)/\Delta(z)]^{\frac{1}{4}} = [\tfrac{1}{4}\, d\zeta/dz]^{-\frac{1}{2}},$$

the fractional powers being determined so that $G(z)$ is real and positive in both axes within the above strip. The basic equation is now III, (1.9) with the negative sign, and the basic functions are modified parabolic cylinder functions.

Provided λ' satisfies (3.5*a*), there are paths similar to those introduced in §3.1 above, originating from ∞i or from $\frac{1}{2}\pi + \infty i$ and terminating in an arbitrary point of Ω, for which the estimate (3.6*a*) is valid if $\sin z$ is replaced by $\cos z$. No new problems arise in connection with the formulae (3.27*a*, *b*) for $y_1(z)$, $y_1(z-\pi)$. For the phase parameter Φ, the formula for $y_1(-z)$ obtained from (3.27*b*) by substituting $\pi - z$ for z and $-\zeta$ for ζ is needed; in the remainder, $\eta = h^{-1}\,O(1)$ on Ω.

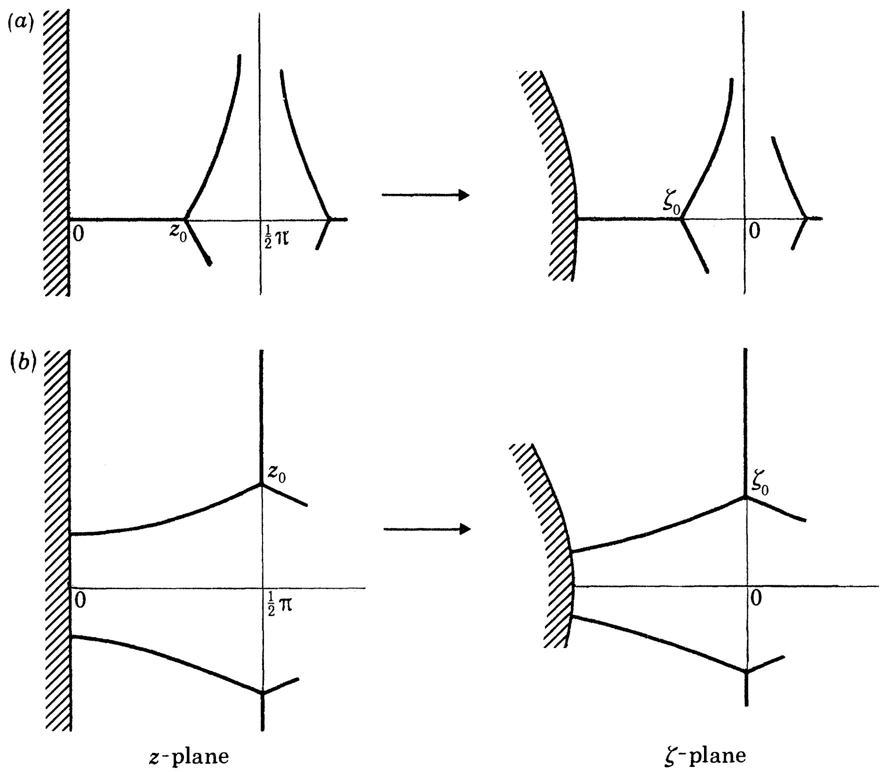

FIGURE 2. The map $z \to \zeta$; $0 \leqslant \lambda' \leqslant 4h^2$; (*a*) $\lambda' < 2h^2$; (*b*) $\lambda' > -2h^2$.

For $y_3(\frac{1}{2}\pi \pm x)$ and $y_4(\pm x)$ as defined in II, (4.3.6), (4.3.17), a similar formula for $y_1(\pi - z)$ is also needed, as well as the connection formulae (3.15*a*, *b*) for the modified parabolic cylinder functions $W(a, \pm x)$. The only feature of the calculations that seems to call for comment is that the remainder term of the formula (3.25) for Φ introduces a term $O(h^{-1})$ into η; this term does not affect the remainder estimate for y_3, but for y_4 it means that the estimate (3.28) is not applicable.

3.3. *Properties of parabolic cylinder functions*

The notation used is that of Miller (1952) and the formulae are taken with some adaptation from Abramowitz & Stegun (1964), except that the second function $\bar{U}$ used here does not appear in those tables; the relation between $\bar{U}$ and the alternative solution V are taken from Miller (1952). Symbols used for variables and parameters are local to this section.

(*a*) *Ordinary functions*

These satisfy the equation $$y'' = (\tfrac{1}{4}z^2 + a)\,y,$$

where the parameter a may be real or complex. A solution characterized by its asymptotic behaviour as $|z| \to \infty$ with $$\arg z = \theta \quad (|\theta| \leqslant \tfrac{1}{4}\pi)$$

is $$U(a, z) \sim \exp(-\tfrac{1}{4}z^2)\, z^{-a-\frac{1}{2}}, \tag{3.10}$$

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this formula remaining valid if $|z| \to \infty$ with $|\arg z| \leqslant \frac{3}{4}\pi - \delta$ ($\delta > 0$); other solutions are $U(a, -z)$, $U(-a, \pm iz)$.

An essentially different solution $\overline{U}(a, z)$ is defined by (3.11 *b*) below, except when $a - \frac{1}{2}$ is a positive integer or zero. If a is real and negative, $U(a, x)$ and $\overline{U}(a, x)$ are oscillatory on the real interval $\{x: |x| < 2|a|^{\frac{1}{2}}\}$, with the same amplitude and with phase differing by $\frac{1}{2}\pi$, while if $x > 2|a|^{\frac{1}{2}}$ they are exponentially decreasing and increasing respectively.

Two connection formulae are

$$U(a, z) = (2\pi)^{-\frac{1}{2}}\, \Gamma(\tfrac{1}{2} - a)\, \{e^{-i\pi(\frac{1}{2}a + \frac{1}{4})}\, U(-a, iz) + e^{i\pi(\frac{1}{2}a + \frac{1}{4})}\, U(-a, -iz)\}, \tag{3.11a}$$

$$\overline{U}(a, z) = (2\pi)^{-\frac{1}{2}}\, \Gamma(\tfrac{1}{2} - a)\, \{e^{-i\pi(\frac{1}{2}a - \frac{1}{4})}\, U(-a, iz) + e^{i\pi(\frac{1}{2}a - \frac{1}{4})}\, U(-a, -iz)\}. \tag{3.11b}$$

Let the parameter a be real and define

$$t = \int (\tfrac{1}{4}z^2 + a)^{\frac{1}{2}}\, dz, \tag{3.12}$$

with the branch and particular integral determined on the plane cut along the interval $[-2(-a)^{\frac{1}{2}}, 2(-a)^{\frac{1}{2}}]$ by the conditions

$$\left.\begin{array}{ll} \text{(i)} & t \text{ is real and positive if } \arg z = 0 \text{ and } z^2 + 4a > 0, \\ \text{(ii)} & \operatorname{Re} t = 0 \quad \text{when} \quad z = 0. \end{array}\right\} \tag{3.12a}$$

Then

$$U(a, z) \sim e^{\frac{1}{4}a} |a|^{-\frac{1}{4}a}\, (z^2 + 4a)^{-\frac{1}{4}}\, e^{-t} \tag{3.13}$$

as $z \to \infty$ with $|\arg z| \leqslant \frac{3}{4}\pi - \delta$ ($\delta > 0$), the fractional power being real and positive as $z \to \infty$ with $\arg z = 0$.

This formula is obtained by the L.–G. method; to determine the factors depending only on a, (3.12) is integrated explicitly, giving

$$t = \tfrac{1}{4}z^2 + \tfrac{1}{2}a + a \ln z - \tfrac{1}{2}a \ln |a| + O(z^{-2});$$

this is substituted into (3.13) and the result compared with (3.10).

(*b*) *Modified functions with real parameter*

These satisfy the equation $\qquad y'' = (-\frac{1}{4}z^2 + a)\, y,$

where a is real, which has solutions

$$U(\mp ia, z e^{\pm\frac{1}{4}i\pi}), \quad U(\mp ia, z e^{\mp\frac{3}{4}i\pi}).$$

The connection formula (3.11 *a*) becomes

$$U(ia, z e^{-\frac{1}{4}i\pi}) = (2\pi)^{-\frac{1}{2}}\, \Gamma(\tfrac{1}{2} - ia) \times \{e^{-\frac{1}{4}i\pi} e^{\frac{1}{2}\pi a}\, U(-ia, z e^{\frac{1}{4}i\pi}) + e^{\frac{1}{4}i\pi} e^{-\frac{1}{2}\pi a}\, U(-ia, -z e^{\frac{1}{4}i\pi})\}, \tag{3.14}$$

with a similar complex conjugate formula.

For real variable x, a real solution introduced by Miller is

$$W(a, x) = (\tfrac{1}{2}k)^{\frac{1}{2}}\, e^{\frac{1}{4}i\pi a}\{e^{\frac{1}{8}i\pi + \frac{1}{2}i\phi_2}\, U(ia, x e^{-\frac{1}{4}i\pi}) + e^{-\frac{1}{8}i\pi - \frac{1}{2}i\phi_2}\, U(-ia, x e^{\frac{1}{4}i\pi})\}, \tag{3.15}$$

when

$$W(a, -x) = -i(2k)^{-\frac{1}{2}}\, e^{\frac{1}{4}\pi a}\{e^{\frac{1}{8}i\pi + \frac{1}{2}i\phi_2}\, U(ia, x e^{-\frac{1}{4}i\pi}) - e^{-\frac{1}{8}i\pi - \frac{1}{2}i\phi_2}\, U(-ia, x e^{\frac{1}{4}i\pi})\}, \tag{3.15a}$$

where

$$k^{\pm 1} = \sqrt{(1 + e^{2\pi a})} \mp e^{\pi a}$$

and

$$\phi_2 = \arg \Gamma(\tfrac{1}{2} + ia). \tag{3.15b}$$

If $a > 0$ and $|x| < 2a^{\frac{1}{2}}$, these solutions are exponential in behaviour, the former being decreasing; otherwise they are oscillatory with phase differing by $\frac{1}{2}\pi$.

Let t be defined by (3.12) with conditions (3.12a), but with $-a$ substituted for a throughout. Then the formula corresponding to (3.13) is:

$$U(\mathrm{i}a,\ z\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}) \sim \mathrm{e}^{\frac{1}{8}\mathrm{i}\pi}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}a}|a|^{-\frac{1}{2}\mathrm{i}a}\,\mathrm{e}^{-\frac{1}{4}\pi a}(z^2-4a)^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}t} \tag{3.16}$$

as

$$z \to \infty \quad \text{with} \quad -\tfrac{1}{2}\pi+\delta \leqslant \arg z \leqslant \pi-\delta \quad (\delta > 0),$$

the fourth root being real and positive as $z \to \infty$ with $\arg z = 0$. The formula obtained by substituting $-\mathrm{i}$ for i is valid on the complex conjugate domain.

3.4. *Table of asymptotic formulae*

For definitions of symbols not defined here, and in particular the interpretation of the symbol $O(1)$, see part II, §4 and part IV, §6. Throughout this section,

$$\lambda'-\lambda = \tfrac{1}{8}.$$

3.4.1. *The case* $-4h^2 \leqslant \lambda' \leqslant 0$

(*a*) *Auxiliary parameters*

Define

$$\kappa = 2hE/\pi,$$

$$\sigma = (2\pi)^{\frac{1}{2}}\,(|\kappa|/\mathrm{e})^{|\kappa|}\,[\Gamma(|\kappa|+\tfrac{1}{2})]^{-1}. \tag{3.17}$$

Then

$$2^{-\frac{1}{2}} \leqslant \sigma < 1$$

and

$$0 < \ln\sigma + |(24\kappa)|^{-1} < |(360\kappa^3)|^{-1}.$$

(i) If $\lambda' \leqslant -2h^2$,

$$2\cosh(\pi\mu) = \sigma\,\mathrm{e}^{2hE_1}[1+O(h^{-1})] \tag{3.18}$$

and Φ satisfies IV, (6.2.2);

(ii) If $\lambda' \geqslant -2h^2$,

$$\hat{\beta} = \sigma\,\mathrm{e}^{-2hE_1}[1+O(h^{-1})] \tag{3.19}$$

and Φ satisfies IV, (6.2.4).

(*b*) *Complex variable*

On the strip $\{z\colon |\mathrm{Re}\,z| \leqslant \frac{1}{2}\pi\}$, define the variable ζ by

(i) $$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = 2\left\{\frac{\lambda'/h^2+2\cos 2z}{\alpha-\zeta^2}\right\}^{\frac{1}{2}},$$

where $\alpha = 8E/\pi = 4\kappa/h$ is real, and is positive (negative) if $\lambda' > (<) -2h^2$;

(ii) $\zeta = 0$ and $\mathrm{d}\zeta/\mathrm{d}z$ is real and positive when $z = 0$.

The map $z \to \zeta$ is one-to-one analytic on the strip and is shown in figure 1 (§3.1).

Define also the transformation factor

$$G(z) = [\tfrac{1}{4}\mathrm{d}\zeta/\mathrm{d}z]^{-\frac{1}{2}},$$

real and positive on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and on the imaginary axis.

In the following asymptotic formulae,

$$\Theta = h^{\frac{1}{4}}|G(z)|\,\min\{|h\zeta^2-4\kappa|^{-\frac{1}{4}},\ \kappa^{-\frac{1}{12}},\ 1\}.$$

Then on Ω,

$$y_1(z) = \mathrm{e}^{-hE_1}(|\kappa|/\mathrm{e})^{\frac{1}{2}\kappa}\,h^{\frac{1}{4}}G(z)\ U(\kappa,\ -\mathrm{i}h^{\frac{1}{2}}\zeta)+\Theta\,\mathrm{e}^{h\xi}\,\eta, \tag{3.20a}$$

$$y_1(z-\pi) = \mathrm{e}^{-\mathrm{i}\pi(\frac{1}{4}+\frac{1}{2}\kappa)}\,\mathrm{e}^{hE_1}(|\kappa|/\mathrm{e})^{-\frac{1}{2}\kappa}h^{\frac{1}{4}}G(z)\ U(-\kappa,\ h^{\frac{1}{2}}\zeta)+\Theta\,\mathrm{e}^{-h\xi}\,\eta, \tag{3.20b}$$

where, uniformly,

$$\eta = \begin{cases} [\Delta(z)]^{-\frac{1}{4}}\, h^{-1}O(1) & \text{if} \quad |\sin z| \geqslant 1 \\ h^{-1}O(1) & \text{otherwise.} \end{cases} \tag{3.21}$$

(*c*) *Modified functions*, $q < 0$

If $x \geqslant 0$, with $z = \mathrm{i}x$:

(i) if $\kappa \leqslant 0$, $y_1(\mathrm{i}x)$ is given by (3.20*a*) and

$$y_2(\mathrm{i}x) = (2\sinh \pi\mu)^{-1}\,\{\mathrm{e}^{hE_1}\sigma(|\kappa|/\mathrm{e})^{\frac{1}{2}\kappa}\, h^{\frac{1}{4}}G(z)\, \overline{U}(\kappa, -\mathrm{i}h^{\frac{1}{2}}\zeta) + \Theta\,\mathrm{e}^{-h\xi}\,\eta\}, \tag{3.22}$$

where η satisfies (3.21);

(ii) if $\kappa \geqslant 0$, $y_1(\mathrm{i}x)$ is again given by (3.20*a*) while $y_1(-\mathrm{i}x)$ is obtained by substituting $-\zeta$ for ζ in (3.20*a*), but with remainder

$$\mathrm{e}^{2hE_1}\Theta\,\mathrm{e}^{-h\xi}\, h^{-1}\, O(1).$$

(*d*) *Ordinary functions*, $q < 0$

If $x \in [0, \frac{1}{2}\pi]$, with $z = x$:

(i) if $\kappa \leqslant 0$,

$$\mathrm{me}\,(\pm x) = (\sinh \pi\mu)^{-1}\, \mathrm{e}^{hE_1}(|\kappa|/\mathrm{e})^{-\frac{1}{2}\kappa}h^{\frac{1}{4}}G(z)\; U(-\kappa, \mp h^{\frac{1}{2}}\zeta) + \mathrm{e}^{-hE_1}\Theta\,\mathrm{e}^{\pm h(\xi+E_1)}\,\eta; \tag{3.23}$$

(ii) if $\kappa \geqslant 0$,

$$y_3(x) = \tfrac{1}{2}\mathrm{e}^{-hE_1}\sigma(\kappa/\mathrm{e})^{-\frac{1}{2}\kappa}h^{\frac{1}{4}}G(z)\; \overline{U}(-\kappa, h^{\frac{1}{2}}\zeta) + \Theta\,\mathrm{e}^{h\xi}\,\eta, \tag{3.24a}$$

$$y_3(\pi - x) = \mathrm{e}^{hE_1}(\kappa/\mathrm{e})^{-\frac{1}{2}\kappa}h^{\frac{1}{4}}G(z)\; U(-\kappa, h^{\frac{1}{2}}\zeta) + \Theta\,\mathrm{e}^{-h\xi}\,\eta, \tag{3.24b}$$

where in each case $\eta = h^{-1}O(1)$ uniformly.

(*e*) *Ordinary and modified functions*, $q > 0$

The former are obtained from (*d*) above by substituting $\frac{1}{2}\pi - x$ for x, while for the latter, the formulae given in part IV, §§6.3(*e*), 6.4(*e*) have satisfactory remainder estimates.

3.4.2. *The case* $0 \leqslant \lambda' \leqslant 4h^2$

(*a*) *Auxiliary parameters*

Define

$$\kappa_1 = 2hE_1/\pi,$$

$$\sigma_1 = (2\pi)^{\frac{1}{2}}\,(\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\kappa_1/\mathrm{e})^{\mathrm{i}\kappa_1}\,[\Gamma(\tfrac{1}{2} + \mathrm{i}\kappa_1)]^{-1}. \tag{3.25}$$

Then $\arg \sigma_1 \sim -(24\kappa_1)^{-1}$ as $\kappa_1 \to \pm\infty$, $\operatorname{sgn} \arg \sigma_1 = -\operatorname{sgn} \kappa_1$

and $0 \leqslant |\arg \sigma_1| < \min\{(22|\kappa_1|)^{-1}, 0.022\}$;

β is given by IV, (6.2.3) and

$$\Phi = \tfrac{1}{4}\pi + \tfrac{1}{2}\pi\kappa_1 - \tfrac{1}{2}\arg \sigma_1 + O(h^{-1}) \quad \text{uniformly.} \tag{3.26}$$

(*b*) *Complex variable*

On the strip $\{z: 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\pi\}$ define the variable ζ by

(i) $$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = 2\left\{\frac{\lambda'/h^2 + 2\cos 2z}{\zeta^2 - \alpha}\right\}^{\frac{1}{2}},$$

where $\alpha = 8E_1/\pi = 4\kappa_1/h$ is real, and is positive (negative) if $\lambda' < (>)\; 2h^2$;

(ii) $\zeta = 0$ and $\mathrm{d}\zeta/\mathrm{d}z$ is real and positive when $z = 0$. The map $z \to \zeta$ is one-to-one analytic on the strip and is shown in figure 2.

Define also the transformation factor

$$G(z) = [\tfrac{1}{4}\mathrm{d}\zeta/\mathrm{d}z]^{-\frac{1}{2}},$$

real and positive on $[0, \pi]$ and on $\{z\colon \operatorname{Re} z = \tfrac{1}{2}\pi\}$.

In the following asymptotic formulae,

$$\Theta = h^{\frac{1}{4}}|G(z)| \min\{|h\zeta^2 - 4\kappa_1|^{-\frac{1}{4}}, \kappa_1^{-\frac{1}{12}}, 1\}.$$

Then on Ω:

$$y_1(z) = \mathrm{e}^{\mathrm{i}\pi(\frac{1}{8}+hE)}\,(|\kappa_1|/\mathrm{e})^{-\frac{1}{2}\mathrm{i}\kappa_1}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi\kappa_1}\,h^{\frac{1}{4}}G(z)\;U(-\mathrm{i}\kappa_1, -\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}h^{\frac{1}{2}}\zeta) + \Theta\,\mathrm{e}^{h\xi}\,\eta, \tag{3.27a}$$

$$y_1(z-\pi) = \mathrm{e}^{-\mathrm{i}\pi(\frac{1}{8}+hE)}\,(|\kappa_1|/\mathrm{e})^{\frac{1}{2}\mathrm{i}\kappa_1}\,\mathrm{e}^{-\frac{1}{2}\pi\kappa_1}h^{\frac{1}{4}}G(z)\;U(\mathrm{i}\kappa_1, \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}h^{\frac{1}{2}}\zeta) + \Theta\cosh[\operatorname{Re}(h\xi)]\,\eta, \tag{3.27b}$$

where uniformly

$$\eta = \begin{cases} [\Delta(z)]^{-\frac{1}{2}}\,h^{-1}O(1) & \text{if } |\cos z| \geqslant 1 \\ h^{-1}O(1) & \text{otherwise.} \end{cases} \tag{3.28}$$

(*c*) *Ordinary functions*, $q > 0$

If $x \in [0, \tfrac{1}{2}\pi]$, with $z = x + \tfrac{1}{2}\pi$,

$$y_3(\tfrac{1}{2}\pi \pm x) = (\tfrac{1}{2}k)^{\frac{1}{2}}\,h^{\frac{1}{4}}G(z)\;W(\kappa_1, \mp h^{\frac{1}{2}}\zeta) + \Theta\,\mathrm{e}^{\pm h\xi}\,\eta, \tag{3.29}$$

where

$$k = (1+\mathrm{e}^{-2\pi\kappa_1})^{\frac{1}{2}} + 1.$$

(*d*) *Modified functions*, $q > 0$

If $x \geqslant 0$, with $z = \tfrac{1}{2}\pi + \mathrm{i}x$,

$$y_4(\pm x) = (\tfrac{1}{2}k_1)^{\frac{1}{2}}\,h^{\frac{1}{4}}G(z)\;W(-\kappa_1, \mp \mathrm{i}h^{\frac{1}{2}}\zeta) + \mathrm{e}^{h\xi_1}\,\Theta\,\mathrm{e}^{\pm h(\xi-\xi_1)}h^{-1}O(1), \tag{3.30}$$

where

$$k_1 = (1+\mathrm{e}^{-2\pi\kappa_1})^{\frac{1}{2}} + \mathrm{e}^{-\pi\kappa_1}$$

and

$$\xi_1 = \begin{cases} -E_1 & \text{if } \lambda' > 2h^2 \\ 0 & \text{if } \lambda' \leqslant 2h^2 \end{cases}$$

(*e*) *Ordinary and modified equations*, $q < 0$

The former are obtained from (*c*) above by substituting $\tfrac{1}{2}\pi - x$ for x, while for the latter, the formulae given in part IV, §§6.3(*b*), 6.4(*b*) have satisfactory remainder estimates.

4. Approximations in terms of Bessel functions

4.1. *The parameter range* $\lambda' \geqslant 4h^2$

(*a*) *The Liouville transformation*

The basic equation used is III, (1.10) with the positive sign and with $u = h$, so that

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{2[\Delta(z)]^{\frac{1}{2}}}{\alpha(1+\zeta^{-2})^{\frac{1}{2}}}. \tag{4.1}$$

The transition points to be taken account of are at $z = \pm\tfrac{1}{2}\pi + \mathrm{i}a$, to be mapped into $\zeta = \mp\mathrm{i}$ respectively. Hence by III, (1.10*a*),

$$\alpha = \frac{1}{\pi}\int_{-\frac{1}{2}\pi+\mathrm{i}a}^{\frac{1}{2}\pi+\mathrm{i}a} 2[\Delta(z)]^{\frac{1}{2}}\,\mathrm{d}z$$
$$= 2E/\pi. \tag{4.1a}$$

With the correct choice of branch in (4.1), the half-strip $\{z: |\mathrm{Re}\, z| \leqslant \frac{1}{2}\pi, \mathrm{Im}\, z \geqslant -a\}$ maps one-to-one *into* the right half-plane of ζ. Moreover, the map $z \to \zeta$ extends analytically to the half-plane $\{z: \mathrm{Im}\, z \geqslant -a\}$, except for branch points on the frontier where $\mathrm{Re}\, z = (n+\frac{1}{2})\pi$, and ζ^2 has period π; in particular, the half-strip $\{z: |\mathrm{Re}\, z| < \pi, \mathrm{Im}\, z \geqslant -a\}$ maps one-to-one into $\{\zeta: |\arg \zeta| < \pi\}$. Figure 3 illustrates this map, with the principal curves issuing from $z_0 = \pm\frac{1}{2}\pi + \mathrm{i}a$.

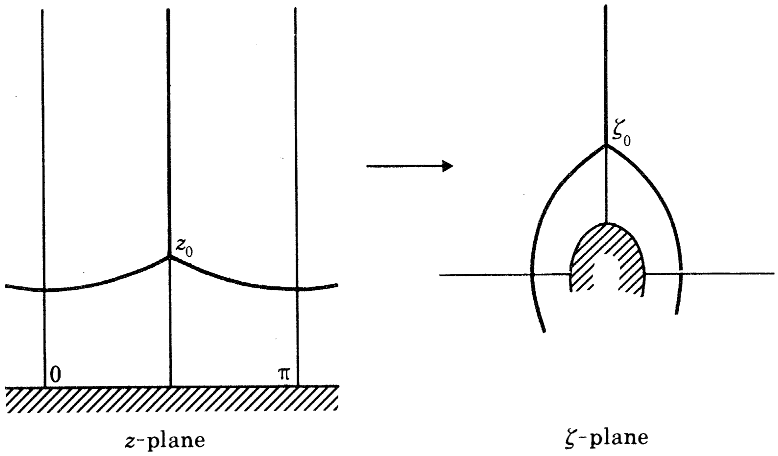

FIGURE 3. The map $z \to \zeta$: $\lambda \geqslant 4h^2$.

The Mathieu equation transforms to

$$\mathrm{d}^2 w/\mathrm{d}\zeta^2 = \{\alpha^2(1+\zeta^{-2})\,[h^2+\psi_3(z)] - \tfrac{1}{4}\zeta^{-2}\}\, w, \tag{4.1b}$$

where $\psi_3(z)$ is given by III, (3.5*a*) and in the latter $\psi(z)$ is given by IV, (1.7*b*), while

$$y = \zeta^{-\frac{1}{2}} G(z)\, w,$$

where
$$G(z) = [-(1+\zeta^2)/\Delta(z)]^{\frac{1}{4}}$$

is analytic and without zeros on $\{z: \mathrm{Im}\, z > -a\}$, and is real and positive if ζ^2 is real.

(*b*) *Estimation of the e.c.f.*

The precise definition of ζ is essential to the effectiveness of the method; nevertheless it can be approximated by the simple formula

$$\alpha\zeta = \mathrm{e}^{a} t[1+\mathrm{e}^{-4a} t^{-2}\, O(1)], \tag{4.2}$$

where $t = \mathrm{e}^{-a-\mathrm{i}z}$, which is uniform on $\{z: \mathrm{Im}\, z \geqslant -a\}$ and over the prescribed parameter range. Indeed, adapting IV, (2.2*a*) one obtains

$$\xi = -\mathrm{e}^{a}\{t - \tfrac{1}{2}(1+\mathrm{e}^{-4a})\, t^{-1} + O(t^{-3})\}$$

as $t \to \infty$; similarly from III, (1.10*b*)

$$\xi = -\alpha\{\zeta - \tfrac{1}{2}\zeta^{-1} + O(\zeta^{-3})\}$$

as $\zeta \to \infty$. In each case the symmetry properties of the map $t \to \xi$ or $\zeta \to \xi$ are used to determine the sign and the constant of integration; the branch of ξ is that defined in part IV, §1.

By comparing these two expressions it can be seen that $\zeta \to \infty$ as $t \to \infty$ and that

$$t(\alpha\zeta - e^{a}t) = O(1) \quad \text{as} \quad t \to \infty,$$

this expression being an analytic function of t on $\{t\colon |t| > e^{-2a}\}$. Next, an explicit integration of III, (1.10b) with $|\arg \zeta| \leqslant \frac{1}{2}\pi$ leads to the estimate

$$\zeta = 2e^{-1-\xi/\alpha}\,[1+o(1)] \quad \text{as} \quad |\zeta| \to 0.$$

But if $\operatorname{Im} z = -a$, then $|t| = e^{-2a}$ and $\operatorname{Re} \xi = -2E_1 > 0$, so with the aid of the estimates $E_1 = -e^{a}[a+O(1)]$ and $\alpha = e^{a}[1+O(e^{-2a})]$, we find

$$|\zeta| = e^{-2a}\,O(1),$$

which leads to the estimate

$$t(\alpha\zeta - e^{a}t) = e^{-a}\,O(1) \quad \text{if} \quad |t| = e^{-2a}.$$

Finally, applying the maximum modulus principle to this function on the domain $\{t\colon |t| \geqslant e^{-2a}\}$, with point at infinity, gives (4.2). As an immediate corollary,

$$\zeta = t[1+O(e^{-2a})]. \tag{4.2a}$$

Similar methods can now be applied to the estimation of $\psi_3(z)$. From (4.2) and IV, (2.6),

$$\psi(z) \sim (\lambda - \lambda' - \tfrac{1}{4})\,e^{2iz} \sim (\lambda - \lambda' - \tfrac{1}{4})\,\alpha^{-2}\zeta^{-2}$$

as $\zeta \to \infty$. But the second term in III, (3.5a) is asymptotically equal to $\frac{1}{4}\alpha^{-2}\zeta^{-2}$ as $\zeta \to \infty$, whence *provided that* $\lambda' - \lambda = 0$, since $\psi_3(z)$ is even and analytic as a function of t or of ζ,

$$\psi_3(z)\,\zeta^4 = O(1) \quad \text{as} \quad \zeta \to \infty.$$

Next, if $\operatorname{Im} z = a - \delta$ where $\delta > 0$, and hence also by symmetry if $\operatorname{Im} z = -a+\delta$, it follows from IV, (2.7) and IV, (2.1) (see also IV, §5.1) that $\psi(z) = e^{-2a}\,O(1)$. But with $\operatorname{Im} z = -a+\delta$, the second term in III (3.5a) is $e^{-6a}\,O(1)$, so that $\psi_3(z) = e^{-2a}\,O(1)$. Thus, by a similar application of the maximum modulus principle,

$$\psi_3(z)\,\zeta^4(1+\zeta^{-2}) = e^{-6a}\,O(1)$$

uniformly on $\{z\colon \operatorname{Im} z \geqslant -a+\delta\}$, this function being analytic on $\{t\colon |t| > e^{-2a}\}$ and at infinity.

Hence

$$\begin{aligned} \psi_3(z)\,d\xi/d\zeta &= \psi_3(z)\,\alpha(1+\zeta^{-2})^{\frac{1}{2}} \\ &= e^{-5a}(1+\zeta^{-2})^{-\frac{1}{2}}\,\zeta^{-4}\,O(1) \end{aligned} \tag{4.3}$$

uniformly on the same region. Integrating this shows that for paths γ originating from infinity in the t-plane with terminal point z such that $\operatorname{Im} z \geqslant -a+\delta$ and satisfying the conditions of lemma 1 of part III, §5 and its corollary,

$$\operatorname*{var}_{\gamma} \int \psi_3(z)\,d\xi = e^{-5a}\,O(1) \times \begin{cases} \zeta^{-3} & \text{if} \quad |\zeta| \geqslant 1 \\ \zeta^{-2} & \text{if} \quad |\zeta| \leqslant 1. \end{cases} \tag{4.3a}$$

By (4.2a) above, ζ may be replaced by t in this formula.

Figure 1 of part III and the accompanying text indicate the region accessible from ∞i in the z-plane by ξ-progressive paths, but the present analysis only applies to that part of the region on which $\operatorname{Im} z \geqslant -a+\delta$, which in particular excludes the transition points in the lower half-plane. Provided $\operatorname{Re} z$, or equivalently $\arg \zeta$, is uniformly bounded, paths consisting of an unbounded initial arc on which Im ξ is constant, followed by a finite arc on which Re ξ is constant, satisfy the necessary conditions. The corresponding paths originating from $\pi+\infty i$ and the region of accessibility are obtained by displacement through π.

(*c*) *Derivation of asymptotic formulae*

The basic functions have the form

$$(\kappa\zeta)^{\frac{1}{2}}\,\mathscr{Z}_{\kappa}(\kappa\zeta),$$

where $\kappa = h\alpha = 2hE/\pi$, and $\mathscr{Z}_{\kappa}$ denotes a modified Bessel function of order κ. The formula (4.15*a*) follows readily with the aid of IV, (1.9) and (4.8*a*) below; the remainder estimate is obtained from (4.3*a*) above and an estimate for κ. The domain of validity includes the half-strip $\{z\colon |\mathrm{Re}\, z| \leqslant \pi,\ \mathrm{Im}\, z \geqslant 0\}$. To obtain (4.15*b*) substitute $z-\pi$ for z and $\zeta e^{i\pi}$ for ζ.

The remainder estimate shows a substantial improvement over IV, (6.3.2) which applies to the corresponding formulae of IV, §6.5, both for large and for small values of Im z, and not merely in the neighbourhood of the transition point. However the factor $O(1)$ is not uniform with respect to κ unless κ is bounded away from zero; this is because the factor $O(1)$ in III, (2.3*a*) relating to the form of the majorants, and hence in turn the constant c in III (1.5*a*), would otherwise not be uniform.

The problem now is to obtain a formula for $y_1(-z)$ valid on the same region; this may be done as follows. Comparison of (4.9*a*) below, κ, ζ being substituted for ν, x and correspondingly $-h\xi$ for νt, with IV, (6.4.1*c*), which is valid when $\lambda \geqslant 4h^2$, suggests the construction of a solution

$$y(z) = \kappa^{\frac{1}{2}}G(z)\,\{I_{\kappa}(\kappa\zeta)+\boldsymbol{I}_{\kappa}\,\eta\} \tag{4.4}$$

corresponding to the basic function $(\kappa\zeta)^{\frac{1}{2}}I_{\kappa}(\kappa\zeta)$. The required paths are $-\xi$-progressive, and if the initial point is on the real axis of z, the accessible domain is the complete upper half-plane; by means of (4.3*a*) with ζ taken at the *initial* point,

$$\eta = \kappa^{-1}e^{-2a}\,O(1).$$

If, now, exponential approximations can be obtained for this solution and for $y_1(\pm z)$, valid on $[-\pi, \pi]$ and with satisfactory remainder estimates, the method of III, §4 can be used to express $y_1(-z)$ in terms of the other two solutions.

Therefore let $z \in [-\pi, \pi]$ so that $|t| = e^{-a}$ and $|\arg t| \leqslant \pi$. Then it follows immediately from (4.9*a*) below that

$$y(z) = e^{\frac{1}{4}i\pi}(2\pi)^{-\frac{1}{2}}\,(\sigma^*)^{-1}\,F(z)\,e^{-h\xi}[1+\kappa^{-1}e^{-2a}\,O(1)].$$

In the same way, from (4.15*a*) and (4.9*b*) (which see for the definitions of σ^*, ϵ_1),

$$y_1(z) = e^{\frac{1}{4}i\pi}\sigma^*F(z)\,e^{h\xi}[1+\epsilon_1 O(1)],$$

where in addition to the previous substitutions, e^{-a} is substituted for x in the formula for ϵ_1; this is justified by (4.2*a*). Substituting $-z$ for z and in consequence $-\xi-2E_1$ for ξ (compare the method of IV, §4 (*a*)),

$$y_1(-z) = e^{\frac{1}{4}i\pi}\sigma^*\,e^{-2hE_1}\,F(z)\,e^{-h\xi}[1+\epsilon_1 O(1)].$$

From these formulae it follows by III, §4 that

$$y_1(-z) = (2\pi)^{\frac{1}{2}}\,(\sigma^*)^2\,e^{-2hE_1}\,[1+\epsilon_1 O(1)]\,y(z)+\epsilon_1 O(1)\,y_1(z). \tag{4.5}$$

From this and (4.4), (4.15 c) follows and is valid on the above half-strip; by comparing ϵ_1 with the estimate for η in (4.4) and also by means of (4.15 a) and the asymptotic formulae (4.8 a, b), it can be shown that the terms involving ϵ_1 in both coefficients in (4.5) can be absorbed into the final remainder term. The formulae (4.14 a, b) for $\hat{\beta}$, Φ are now found from (4.7 b) and II, (4.3.2) by using (4.15 a, b, c).

Finally, for $y_4(\pm x)$, (4.16 a) is constructed by means of the defining formula II, (4.3.17) and (4.7 c) below, and (4.16 b) is obtained similarly from (4.7 d) and II, (3.13). However the latter is not valid with the remainder estimate given if $x < a$, since $y_4(-x)$ is recessive as x decreases on this interval. Instead (4.15 c), and a formula for $y_1(\pi - z)$ obtained from it by the appropriate substitutions, are substituted into II, (4.3.17) with $\pi - z$ in place of z in this last; replacing ϕ by zero gives (4.17 c) without affecting the remainder estimate.

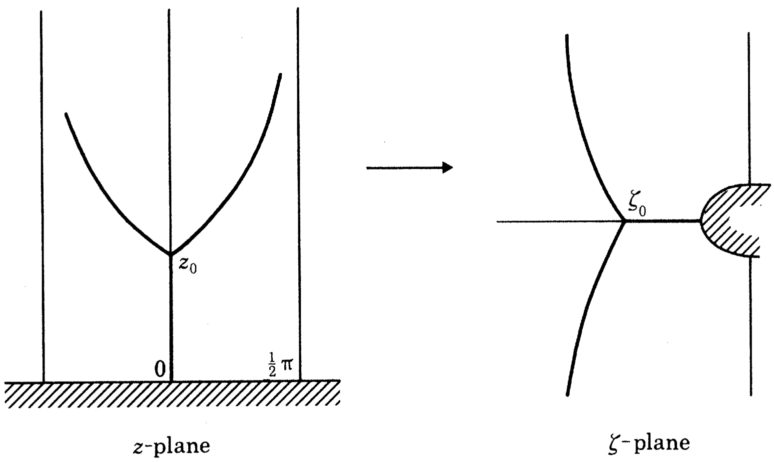

FIGURE 4. The map $z \to \zeta$: $\lambda \leqslant -4h^2$.

4.2. *The parameter range* $\lambda' \leqslant -4h^2$

The basic equation is now III, (1.10) with the negative sign, so that

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{2[\Delta(z)]^{\frac{1}{2}}}{\alpha(1-\zeta^{-2})^{\frac{1}{2}}},$$

where

$$\alpha = \frac{1}{\pi}\int_{\mathrm{i}a}^{\pi+\mathrm{i}a} 2[\Delta(z)]^{\frac{1}{2}}\,\mathrm{d}z = \frac{2E_1}{\pi}.$$

With the correct branch and constant of integration, the half-strip $\{z: |\mathrm{Re}\, z| < \pi, \mathrm{Im}\, z \geqslant -a\}$ again maps one-to-one *into* $\{\zeta: |\arg \zeta| < \pi\}$, but this time with $z = \mathrm{i}a$, $\mp\pi + \mathrm{i}a$ mapping into $\zeta = 1$, $\mathrm{e}^{\pm \mathrm{i}\pi}$ respectively. The map and principal curves are illustrated in figure 4.

The Mathieu equation now transforms to

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{\alpha^2(1-\zeta^{-2})\,[h^2+\psi_3(z)] - \tfrac{1}{4}\zeta^{-2}\}\,w,$$

with the corresponding expression III, (3.5 a) for $\psi_3(z)$, while

$$y = \zeta^{-\frac{1}{2}}G(z)\,w$$

as before, but

$$G(z) = [-(1-\zeta^2)/\Delta(z)]^{\frac{1}{4}}$$

which is again analytic on $\{z: \text{Im } z \geqslant -a\}$ and is real and positive on both axes in the ζ-plane. The expression (4.2) for ζ in terms of $t = e^{-a-iz}$, and the estimate (4.3 a) for the variation of the e.c.f., remain valid.

The formula (4.18 a), where $\kappa_1 = 2hE_1/\pi$, is readily established. The whole of the half-plane $\{z: \text{Im } z \geqslant a\}$ is accessible by ξ-progressive paths originating from ∞i, and if $|\zeta| \leqslant 1$, the only restriction required for the validity of the remainder estimate (4.19) is that $|\text{Re } z|$ should be bounded. However, if $|\zeta| > 1$ and $|\text{Re } z| > \pi$, the condition of the corollary of lemma 1 of part III cannot be satisfied with z^* as the terminal point of the path, and (4.19) must be replaced by

$$\eta = \kappa^{-1} e^{-4a} O(1).$$

For the reason indicated in §4.1 above in relation to κ, it is always necessary that κ_1 should be bounded away from zero.

By substituting $\pi - z$ for z and $\zeta e^{i\pi}$ for ζ in (4.18 a), (4.18 b) is obtained. Similarly,

$$y_1(z+\pi) = (2\kappa_1/\pi)^{\frac{1}{2}} G(z) K_{i\kappa_1}(\kappa_1 \zeta e^{-i\pi}) [1 + \kappa_1^{-1} e^{-4a} O(1)] \tag{4.6}$$

as $z \to \pi + \infty i$. The connection formulae II, (4.1.2) and (4.10 a) below now give

$$\cosh (\pi\mu) = \cosh (\pi\kappa_1) [1 + \kappa_1^{-1} e^{-4a} O(1)],$$

from which (4.17 b) follows.

The estimation of the phase parameter Φ is more difficult; it can be combined with the construction of a formula for a characteristic solution. By II, (3.14), with (4.18 b), (4.6) and (4.17 a),

$$\text{Me}_+(z) = (2\kappa_1/\pi)^{\frac{1}{2}} (\sinh 2\pi\mu)^{-1} G(z) \{e^{\pi\kappa_1}[K_{i\kappa_1}(\kappa_1 \zeta e^{-i\pi}) + \boldsymbol{K}\, \eta_1] - e^{-\pi\kappa_1}[K_{i\kappa_1}(\kappa_1 \zeta e^{i\pi}) + \boldsymbol{K}\, \eta_2]\},$$

where
$$\eta_1, \eta_2 = \begin{cases} \kappa_1^{-1} e^{-4a} O(1) & \text{if} \quad |\zeta| \geqslant 1, \\ \kappa_1^{-1} e^{-4a} \zeta^{-2} O(1) & \text{if} \quad |\zeta| \leqslant 1. \end{cases}$$

Now by (4.10 b) the principal terms in the expression in curly brackets combine to give

$$2\pi \cosh (\pi\kappa_1) I_{i\kappa_1}(\kappa_1 \zeta),$$

and it is found that the contributions from the majorant functions combine correspondingly. Using the formula above for cosh $(\pi\mu)$ gives the result

$$\text{Me}_+(z) = i(\sinh \pi\mu)^{-1} (2\kappa_1 \pi)^{\frac{1}{2}} G(z) \{I_{i\kappa_1}(\kappa_1 \zeta) + \boldsymbol{I}\, \eta\}$$

with the above formula for η.

The next step is to derive from this, by means of (4.12 a), the following formula in terms of $e^{h\xi}$, valid on a bounded interval in the real axis of z, ξ being defined by continuation from $[0, \frac{1}{2}\pi]$:

$$\text{Me}_+(z) = e^{\frac{1}{4}i\pi}(\sinh \pi\mu)^{-1} e^{\pi\kappa_1}(\sigma_1^*)^{-1} F(z) e^{h\xi}[1 + \kappa_1^{-1} e^{-2a} O(1)].$$

Since on the real axis of z, $\text{Im } \xi = E$, and $\Phi = \arg \text{Me}_+(z)$, Φ is given by (4.17 c). The remaining formulae of §4.4 (b) present no new difficulties.

4.3. *Properties of Bessel functions*

Notation is standard, and the formulae are taken with some adaptation from Abramowitz & Stegun (1964), with the exception of (4.9 b), which appears to be new. Symbols used for variables and parameters are local to this section.

(*a*) *Ordinary and modified functions of real order* ν

(i) *Connection formulae:*

$$2\cos(\pi\nu)\,K_\nu(x) = K_\nu(x\,e^{-i\pi}) + K_\nu(x\,e^{i\pi}), \tag{4.7a}$$

$$I_\nu(x) = (\pi i)^{-1}\{e^{-i\pi\nu}K_\nu(x) - K_\nu(x\,e^{i\pi})\}, \tag{4.7b}$$

$$\left.\begin{aligned} J_\nu(x) &= (\pi i)^{-1}\{e^{-\frac{1}{2}i\pi\nu}K_\nu(x\,e^{-\frac{1}{2}i\pi}) - e^{\frac{1}{2}i\pi\nu}K_\nu(x\,e^{\frac{1}{2}i\pi})\},\\ Y_\nu(x) &= -\pi^{-1}\{e^{-\frac{1}{2}i\pi\nu}K_\nu(x\,e^{-\frac{1}{2}i\pi}) + e^{\frac{1}{2}i\pi\nu}K_\nu(x\,e^{\frac{1}{2}i\pi})\},\end{aligned}\right\} \tag{4.7c}$$

$$I_\nu(x\,e^{i\pi}) = e^{i\pi\nu}I_\nu(x); \quad J_\nu(x) = e^{\frac{1}{2}i\pi\nu}I_\nu(x\,e^{-\frac{1}{2}i\pi}). \tag{4.7d}$$

(ii) *Asymptotic formulae:* Define

$$t = \int (1+x^{-2})^{\frac{1}{2}}\,dx,$$

the branch and constant of integration being determined so that on $\{x: |x| > 1\}$, $t = x + O(x^{-1})$; t is then an odd function of x. Then

$$K_\nu(\nu x) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\,\nu^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{4}}\,e^{-\nu t}\,[1+o(1)] \tag{4.8a}$$

as $\quad x \to \infty \quad$ with $\quad |\arg x| \leqslant \tfrac{3}{2}\pi - \delta$

and

$$I_\nu(\nu x) = (2\pi)^{-\frac{1}{2}}\nu^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{4}}\,e^{\nu t}\,[1+o(1)] \tag{4.8b}$$

as $\quad x \to \infty$ with $|\arg x| \leqslant \tfrac{1}{2}\pi - \delta$, where $\delta > 0$.

If $|x| \leqslant \rho < 1$, and t is defined by continuation along the real axis with $\arg x = 0$, then $t = \ln x + O(1)$. Then

$$I_\nu(\nu x) = (2\pi)^{-\frac{1}{2}}\,[\sigma^*]^{-1}\,\nu^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{4}}\,e^{\nu t}\,[1+\nu^{-1}x^2\,O(1)] \tag{4.9a}$$

and

$$K_\nu(\nu x) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\,\sigma^*\nu^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{4}}\,e^{-\nu t}\,(1+\epsilon_1), \tag{4.9b}$$

where

$$\epsilon_1 = \begin{cases} \nu^{-1}x^2[O(1) + \min\{|\ln x|, (\nu-1)^{-1}\}\,O(1)] & \text{if } \nu > 1,\\ \nu^{-1}x^{2\nu}[O(1) + \min\{|\ln x|, (1-\nu)^{-1}\}\,O(1)] & \text{if } \nu < 1,\\ x^2[O(1) + |\ln x|\,O(1)] & \text{if } \nu = 1.\end{cases}$$

In these formulae,

$$\sigma^* = (2\pi)^{-\frac{1}{2}}\,\Gamma(\nu)\,e^{\nu}\nu^{\frac{1}{2}-\nu} = 1 - O(\nu^{-1}); \tag{4.9c}$$

the term and the factor $O(1)$ are uniform on $\{x: |x| \leqslant \rho\}$ and are also uniform with respect to ν provided ν is bounded away from zero.

(*b*) *Modified functions of purely imaginary order* $i\nu$ ($\nu > 0$)

(i) *Connection formulae:*

$$2\cosh(\pi\nu)\,K_{i\nu}(x) = K_{i\nu}(x\,e^{-i\pi}) + K_{i\nu}(x\,e^{i\pi}), \tag{4.10a}$$

$$2\cosh(\pi\nu)\,I_{\pm i\nu}(x) = (\pi i)^{-1}\{e^{\pm\pi\nu}K_{i\nu}(x\,e^{-i\pi}) - e^{\mp\pi\nu}K_{i\nu}(x\,e^{i\pi})\}. \tag{4.10b}$$

(ii) *Asymptotic formulae:*

With

$$t = \int (1-x^{-2})^{\frac{1}{2}}\,dx$$

defined on $\{x: |x| > 1\}$ so that $t = x + O(x^{-1})$ as $x \to \infty$:

$$K_{i\nu}(\nu x) = K_{-i\nu}(\nu x) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, \nu^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\, e^{-\nu t}\,[1+o(1)] \tag{4.11a}$$

as $x \to \infty$ with $|\arg x| \leqslant \tfrac{3}{2}\pi - \delta$

and
$$I_{\pm i\nu}(\nu x) = (2\pi)^{-\frac{1}{2}}\, \nu^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\, e^{\nu t}\,[1+o(1)] \tag{4.11b}$$

as $x \to \infty$ with $|\arg x| \leqslant \tfrac{1}{2}\pi - \delta$, where $\delta > 0$.

Also, when $|x| \leqslant \rho < 1$, the branches of t and of the fractional power being defined by continuation along the imaginary axis with $\arg x = \tfrac{1}{2}\pi$,

$$I_{i\nu}(\nu x) = (2\pi)^{-\frac{1}{2}}\,[\sigma_1^*]^{-1}\, e^{-\frac{1}{4}i\pi}\nu^{-\frac{1}{2}}\, e^{\pi\nu}(1-x^2)^{-\frac{1}{4}}\, e^{-\nu t}\,(1+\eta), \tag{4.12a}$$

where
$$\eta = \nu^{-1}x^2\, O(1)$$

and
$$\sigma_1^* = (2\pi)^{-\frac{1}{2}}\, \Gamma(i\nu)\, e^{i\nu}(e^{\frac{1}{2}i\pi}\nu)^{\frac{1}{2}-i\nu}. \tag{4.12b}$$

Then
$$\sigma_1^* = 1+O(\nu^{-1}) \quad \text{and} \quad |\sigma_1^*| = (1-e^{-2\pi\nu})^{-\frac{1}{2}}.$$

4.4. *Table of asymptotic formulae*

For definitions of symbols not defined here, see part II, §4 and part IV, §6. Throughout this section

$$\lambda' = \lambda.$$

4.4.1. *The case* $\lambda \geqslant 4h^2$

(*a*) *Auxiliary parameters.* Define

$$\left.\begin{aligned} \kappa &= 2hE/\pi, \\ \sigma^* &= (2\pi)^{-\frac{1}{2}}\, e^{\kappa}\kappa^{\frac{1}{2}-\kappa}\Gamma(\kappa). \end{aligned}\right\} \tag{4.13}$$

Then
$$\kappa > 0, \quad \sigma^* = 1+\kappa^{-1}O(1)$$

and
$$\hat{\beta} = [\sigma^*]^2\, e^{-2hE_1}\,(1+\epsilon_1), \tag{4.14a}$$

$$\Phi = \tfrac{1}{4}\pi + hE + \epsilon_2, \tag{4.14b}$$

where

$$\epsilon_1, \epsilon_2 = \begin{cases} \kappa^{-1}e^{-2a}[O(1)+\min\{a, 1/(\kappa-1)\}\,O(1)] & \text{if} \quad \kappa > 1, \\ \kappa^{-1}e^{-2a\kappa}[O(1)+\min\{a, 1/(1-\kappa)\}\,O(1)] & \text{if} \quad \kappa < 1 \\ e^{-2a}[O(1)+a\,O(1)] \quad \text{if} \quad \kappa = 1, \end{cases} \tag{4.14c}$$

provided that κ is bounded away from zero.

(*b*) *Complex variable.* On the region $\{z: \operatorname{Im} z > -a\}$ define the variable ζ by

$$\text{(i)} \quad \frac{d\zeta}{dz} = \frac{\pi}{E}\,[\Delta(z)/(1+\zeta^{-2})]^{\frac{1}{2}};$$

$$\text{(ii)} \quad \zeta = e^{\pm\frac{1}{2}i\pi} \quad \text{when} \quad z = \mp\tfrac{1}{2}\pi + ia.$$

The map $z \to \zeta$ is analytic on the region and is one-to-one if $|\operatorname{Re} z| < \pi$; it is shown in figure 3.

Define also the transformation factor

$$G(z) = [-(\zeta^2+1)/\Delta(z)]^{\frac{1}{4}},$$

real and positive if ζ^2 is real.

In the following asymptotic formulae,

$$\Theta = G(z)\,\min\{|\zeta^2+1|^{-\frac{1}{4}}, \kappa^{\frac{1}{6}}\}.$$

Then on Ω:
$$y_1(z) = (2\kappa/\pi)^{\frac{1}{2}} G(z)\, K_\kappa(\kappa\zeta) + \Theta\, e^{h\xi}\, \eta, \tag{4.15a}$$
$$y_1(z-\pi) = (2\kappa/\pi)^{\frac{1}{2}} G(z)\, K_\kappa(\kappa\zeta e^{i\pi}) + \Theta\, e^{-h\xi}\, \eta, \tag{4.15b}$$
$$y_1(-z) = (2\pi\kappa)^{\frac{1}{2}} [\sigma^*]^2\, e^{-2hE_1} G(z)\, I_\kappa(\kappa\zeta) + e^{-2hE_1}\Theta\, e^{-h\xi}\, \epsilon_1, \tag{4.15c}$$

where uniformly,
$$\eta = \kappa^{-1} e^{-4a}\, O(1) \times \begin{cases} \zeta^{-3} & \text{if } |\zeta| \geqslant 1, \\ \zeta^{-2} & \text{if } |\zeta| \leqslant 1, \end{cases} \tag{4.15d}$$

and ϵ_1 satisfies (4.14*c*), all provided that κ is bounded away from zero. The approximation $\zeta \approx t = e^{a-iz}$ may be used in (4.15*d*) but *not* in (4.15*a*, *b*, *c*). By using also the fact that $\kappa \sim \lambda^{\frac{1}{2}}$ and $e^{2a} \sim \lambda/h^2$ as $\lambda/h^2 \to \infty$, alternative equivalent expressions for η may be obtained. In particular, an estimate which is not explicitly dependent on z and which is therefore weaker is

$$\eta = \lambda^{-\frac{3}{2}} h^2\, O(1).$$

Subject to κ being bounded away from zero, this is valid even if $h \to 0$, as also is (4.15*d*) itself.

(*c*) *Modified functions*, $q < 0$. $y_1(\pm ix)$ are given by (4.15*a*, *c*) above.

(*d*) *Ordinary functions*. The formulae IV, (6.4.5) (see IV, §6.5(*c*)) can be modified by the insertion of a factor σ^*, the factor β^* being removed; η then satisfies the formula (4.14*c*).

(*e*) *Modified functions*, $q > 0$. If $x \geqslant 0$, with $z = \frac{1}{2}\pi + ix$, then

$$y_4(x) = (\tfrac{1}{2}\pi\kappa)^{\frac{1}{2}} G(z)\, \{-\cos\phi Y_\kappa(i\kappa\zeta) + \sin\phi J_\kappa(i\kappa\zeta)\} + \Theta\, e^{h\xi}\, \eta, \tag{4.16a}$$
$$y_4(-x) = [\hat{\beta} + |\beta|]\, (\tfrac{1}{2}\pi\kappa)^{\frac{1}{2}} G(z)\, \{\cos\phi J_\kappa(i\kappa\zeta) + \sin\phi Y_\kappa(i\kappa\zeta)\} + e^{-2hE_1}\Theta\, e^{-h\xi}\, \eta \quad \text{if} \quad x \geqslant a, \tag{4.16b}$$
$$y_4(-x) = e^{-2hE_1}\{(2\pi\kappa)^{\frac{1}{2}} G(z)\, J_\kappa(i\kappa\zeta) + \Theta\, e^{-h\xi}\, \epsilon_1\} \quad \text{if} \quad x < a, \tag{4.16c}$$

where $\phi = \Phi - \frac{1}{4}\pi - hE$, η satisfies (4.21*d*) and ϵ_1 satisfies (4.14*c*).

4.4.2. *The case* $\lambda \leqslant -4h^2$

(*a*) *Auxiliary parameters*. Define

$$\left.\begin{aligned} \kappa_1 &= 2hE_1/\pi, \\ \sigma_1^* &= (2\pi)^{-\frac{1}{2}}\, e^{i\kappa_1}(e^{\frac{1}{2}i\pi}\kappa_1)^{\frac{1}{2}-i\kappa_1}\Gamma(i\kappa_1); \\ \kappa_1 > 0,\ \sigma_1^* &= 1 + \kappa_1^{-1} O(1) \quad \text{and} \quad |\sigma_1^*| = (1 - e^{-2\pi\kappa_1})^{-\frac{1}{2}}. \end{aligned}\right\} \tag{4.17a}$$

Then
$$\mu = \kappa_1 + \kappa_1^{-1} e^{-4a}\, O(1), \tag{4.17b}$$
$$\Phi = \tfrac{1}{4}\pi + hE - \arg\sigma_1^* + \kappa_1^{-1} e^{-2a}\, O(1), \tag{4.17c}$$

provided κ_1 is bounded away from zero.

(*b*) *Complex variable*. On the region $\{z: \operatorname{Im} z > -a\}$ define the variable ζ by

$$\text{(i)}\quad \frac{d\zeta}{dz} = \frac{\pi}{E_1}\left[\frac{\Delta(z)}{(1-\zeta^2)}\right]^{\frac{1}{2}};$$

$$\text{(ii)}\quad \zeta = 1,\ e^{\pm i\pi} \quad \text{when} \quad z = ia,\ \mp\pi + ia.$$

The map $z \to \zeta$ is analytic on the region and is one-to-one if $|\operatorname{Re} z| < \pi$; it is shown in figure 4. Define also the transformation factor

$$G(z) = [(1-\zeta^2)/\Delta(z)]^{\frac{1}{4}},$$

real and positive if ζ^2 is real.

In the following asymptotic formulae,

$$\Theta = G(z)\min\{|1-\zeta^2|^{-\frac{1}{4}}, \kappa_1^{\frac{1}{6}}\}.$$

So on Ω:

$$y_1(z) = (2\kappa_1/\pi)^{\frac{1}{2}}\, G(z)\, K_{i\kappa_1}(\kappa_1\zeta) + \Theta\, e^{h\xi}\,\eta, \tag{4.18a}$$

$$y_1(z-\pi) = (2\kappa_1/\pi)^{\frac{1}{2}}\, G(z)\, K_{i\kappa_1}(\kappa_1\zeta e^{i\pi}) + \Theta\, e^{-h\xi}\,\eta, \tag{4.18b}$$

where, uniformly,

$$\eta = \kappa_1^{-1} e^{-4a} O(1)\times\begin{cases}\zeta^{-3} & \text{if } |\zeta| \geqslant 1,\\ \zeta^{-2} & \text{if } |\zeta| \leqslant 1,\end{cases} \tag{4.19}$$

provided that κ_1 is bounded away from zero. The remarks following (4.15*d*) apply to (4.19) also, with κ_1 in place of κ.

(*c*) *Modified functions*, $q < 0$. If $x \geqslant 0$, with $z = ix$, then $y_1(ix)$ is given by (4.24*a*) and

$$y_2(ix) = (\sinh \pi\mu)^{-1}\{(2\kappa_1/\pi)^{\frac{1}{2}}\, G(z)\, \mathrm{Re}\,[iK_{i\kappa_1}(\kappa_1\zeta e^{i\pi})] + \Theta\, e^{-h\xi}\,\eta\} \tag{4.20}$$

where η is given by (4.25).

(*d*) *Ordinary functions*. The formulae IV, (6.3.5) are valid with the insertion of a factor $|\sigma_1^*|^{-1}$, and with $\eta = \kappa_1^{-1} e^{-2a} O(1)$.

(*e*) *Modified functions*, $q > 0$. If $x \geqslant 0$, with $z = \frac{1}{2}\pi + ix$, then

$$\mathrm{Me}^*(\pm x) = \pm i(\sinh \pi\mu)^{-1}\, e^{\mp i\Phi}(2\pi\kappa_1)^{\frac{1}{2}}\, e^{\pi\kappa_1} G(z)\, J_{\pm i\kappa_1}(\kappa_1\zeta e^{\frac{1}{2}i\pi}) + \Theta\, e^{\pm h\xi}\,\eta. \tag{4.21}$$

References (Part V)

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